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A SOLUTION TO A DEFLECTION BEAM PROBLEM

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ABSTRACT

The present paper shows how the minimal total potential energy principle it is used for elastic systems in the case of the determination of the linear-elastic displacements which determines the beams bending. Qualitative aspects (which usually are hidden) can be studied by displacements functions, relevated by this method which is presented in my work.

1. Preliminary

Usually, the calculus model for the structural mechanical problems contains a set of differential equations which could not be analytically integrated, so it is impossible to find the exact solution to the given problem.

For this case we use a new way to find acceptable approximative solution, giving a satisfactory precision, good enough for practical purposes.

Considering the certain situation of applied elastic theory problems, it could be possible to use the variational methods, which assures a good precision for the obtain solutions.

According with the calculus of variation, a certain differential equation’s solution (for an acceptable variable’s domain) with proper boundary conditions, is equal with the minimization, in the considered domain, of a suitable functional which corresponds to the differential equation and their given boundary conditions.

Lets consider an homogeneous, isotropic, linear-elastic body, which is however supported, loaded with a certain load system. With the acceptance of the assumption that there is no energy dissipative elements and the stress in the body is less then the elastic limit of the material, emphasizes the elasticity of the system. In these conditions, the potential total energy of the system is:

$$\Pi = U - W \tag{1}$$

where U is the internal potential energy for the considered elastic system and W is the work of the external loads on the virtual displacements, related with the bindings.

For the equilibrium of the body, the total potential energy variation is null, so:

$$\delta\Pi = \delta(U - W) = 0 \tag{2}$$

The equation (2) imply that the Π function has for the equilibrium position of the body, one of the extremally value: the minimum value for the stable equilibrium and the maximum value for the instable equilibrium.

All these highlights that *the total potential energy minimum principium* according with, in the stable equilibrium position of the body, *the total potential energy is minimum*.

Because of his generality, this principium can be used for the solution of any equilibrium elastic body problem.

Clearly, we can not renounce at the classical calculus methods, methods which are based on the static equilibrium equations. We can use the energetical methods in the special cases, when it is the simplest way to work on it, or to emphasize some hardly observing aspects.

2. Example study

Considering a simple beam with l length, having an uniform flexural rigidity, loaded on a linear distribution law (fig. 1).

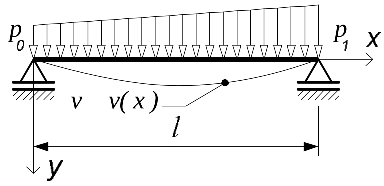

Figure 1.

If the hypothesis considered in the first paragraph are verified, then we can calculate the beam bending deformations, useing the total potential energy principium.

Lets take the elastic line of the beam being described by the function:

$$v(x)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \quad (3)$$

where a_n are some constants which will be determined. The potential deformation energy, stored by the beam is:

$$U=\int_0^l \frac{EI_z}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx \quad (4)$$

Assuming (3), after some elementary calculus, we obtain for the total potential energy stored by the beam, the relation:

$$U=\frac{\pi^4 EI_z}{4l^3}\sum_{n=1}^{\infty} n^4 a_n^2 \quad (5)$$

The work of the external forces, in the considered situation, is given by:

$$W=\int_0^l p(x)v(x)dx \quad (6)$$

where $p(x)$ is the load intensity on the current cross section:

$$p(x)=\frac{p_1-p_0}{l}x+p_0 \quad (7)$$

From (3) and (7) relations, we find:

$$W=\sum_{n=1}^{\infty} a_n\left(\frac{p_1-p_0}{l}J_1+p_0J_2\right) \quad (8)$$

where J_1 and J_2 are:

$$J_1=\int_0^l x\sin\frac{n\pi x}{l}dx=-\frac{l^2}{n\pi}\cos n\pi, \quad (9)$$

respectively:

$$J_2=\int_0^l \sin\frac{n\pi x}{l}dx=\frac{l}{n\pi}(1-\cos n\pi) \quad (10)$$

With (9) and (10), (8) relation become:

$$W=\frac{l}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}(p_0-p_1\cos n\pi) \quad (11)$$

Therefore, from (1) we have:

$$\Pi=\frac{\pi^4 EI_z}{4l^3}\sum_{n=1}^{\infty} n^4 a_n^2-\frac{l}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}(p_0-p_1\cos n\pi) \quad (12)$$

If we imposed the minima condition for the total potential energy:

$$\frac{\partial\Pi}{\partial a_n}=0;\ n=1,2,3,... \quad (13)$$

solving (13) equations system, with unknowns a_n, we find:

$$a_n=\frac{2l^4(p_0-p_1\cos n\pi)}{n^5\pi^5 EI_z} \quad (14)$$

In this conditions the beam's elastic line equation is:

$$v(x)=\frac{2pl^4}{\pi^5 EI_z}\sum_{n=1}^{\infty}\left\{\frac{1}{n^5}(1-\cos n\pi)\sin\frac{n\pi x}{l}\right\} \quad (15)$$

Appling (15) on the beam uniformly distributed load ($p_0=p_1=p$) we may accept that the elastic line's equation is:

$$v(x)\cong 4\frac{pl^4}{\pi^5 EI_z}\sin\frac{\pi x}{l} \quad (16)$$

In this case the maximum deflection appears at the midle of the beam ($x=0{,}5l$) and has the value:

$$v_{max}\cong 0{,}01307\frac{pl^4}{EI_z} \quad (17)$$

which has an verry little error (only $0{,}38\%$!) comparing with the well known theoretical value from the strength of the materials courses (which is find by direct integration of the elastic line's differential equation).

In the same way, for the „triangle" loaded beam ($p_0=p;\ p_1=0$), we find:

$$v(x)\cong 2\frac{pl^4}{\pi^5 EI_z}\sin\frac{\pi x}{l} \quad (18)$$

In this case, by according (18), *the maximum deflection appears at the middle of the beam* (not in the cross section $x=0{,}4807l$, as in the real case) and has the value:

$$v_{max}\cong 0{,}00654\frac{pl^4}{EI_z} \quad (19)$$

which presents an error at only $0{,}31\%$ comparing with well known theoretical value obtained by direct integration.

The qualitative conclusion is that (18) and (19) show that *the displacements of the „triangle" loaded beam are about one half from those the uniformly loaded beam.*

In the same time, for the „triangle" loaded beam too, *the maximum deflection appears in the vicinity of the middle cross section.*

References

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