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STUDY OF THE INTERNAL RESONANCE OF A TWO-DEGREE-OF-FREEDOM NON-LINEAR DYNAMIC SYSTEM

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ABSTRACT

The paper analyses a two-degree-of-freedom dynamic system, by taking into account its non-linearity. The case of the internal resonance is studied and, by local analysis, Hopf bifurcations are emphasized. Periodic motion domains, as well as domains of possible evolution towards chaos have been found, for certain values of the parameters of the system.

1.INTRODUCTION

Two-degrees-of-freedom systems are often described by motion equations that contain second order terms. Such non-linearities result from inertial effects of large motions (oscillations), on one hand, and from Coriolis and centrifugal forces, on the other.

The system has an interesting behavior if the two vibration modes couple also by second order terms. This situation occurs when linear eigenfrequencies of the system are in the ratio 1:2, i.e. in the case known as "internal resonance".

For this type of dynamic systems, complex motions are observed when the frequency of the external perturbation is in the neighborhood of a linear eigenfrequency.

The paper studies, from this point of view, the large oscillations of a mechanical system, which models a vibrating transporter (Fig. 1).

The system consists in:

- the elastically suspended body of mass m_1;
- the electric engine of mass m_2;
- the eccentric mass m_3, with the eccentricity l;
- the elastic springs with the equivalent elasticity constant k;
- the viscous damper of the suspension system with the damping coefficient c_2;
- the hinge O, with the viscous rotation damping c_1.

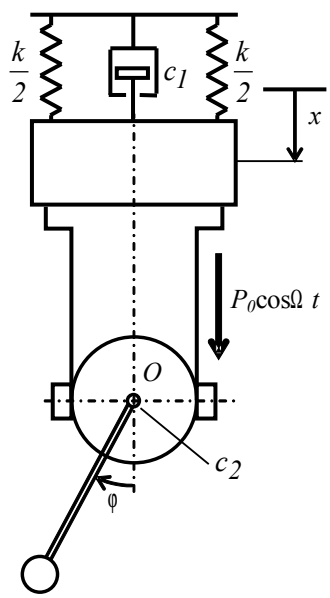


Fig. 1. Mechanical system

2.MODEL AND EQUATIONS OF MOTION

The model of the system and the forces acting upon it are illustrated in Figure 2.

The motion equations are expressed in terms of the two generalized coordinates (Fig. 1):

– the displacement x of the mass center of the body, with respect to the equilibrium position;
– the rotation angle φ, of the eccentric mass.

By denoting the mass of the translating part

$$M = m_1 + m_2 \qquad (0)$$

and by using d'Alembert's principle, the following system is obtained:

$$\begin{cases} (M + m_3)\ddot{x} + c_1\dot{x} + kx - \\ \quad - m_3 l\left(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi\right) = P_0\cos\Omega t \\ m_3 l^2\ddot{\varphi} + c_2\dot{\varphi} + m_3(gl - \ddot{x})\sin\varphi = 0. \end{cases} \qquad (0)$$

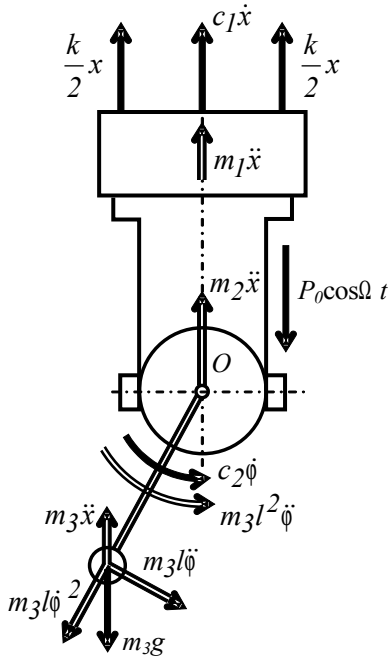


Fig. 2. Forces acting upon the system

This system can be transformed by introducing the notations

$$\begin{cases} \mu = \dfrac{m_3}{M}, \quad F_0 = \dfrac{P_0}{kl} \\ \Omega_1 = \sqrt{\dfrac{k}{M}}, \quad p = \dfrac{\Omega}{\Omega_1} \\ \omega_1 = \sqrt{\dfrac{k}{M + m_3}}, \quad \omega_2 = \sqrt{\dfrac{g}{l}}, \quad q = \dfrac{\omega_2}{\omega_1} \\ 2\alpha_1 = \dfrac{c_1}{M\Omega_1}, \quad 2\alpha_2 = \dfrac{c_2}{m_3 l^2\omega_2} \\ p_1 = \dfrac{1}{p\sqrt{1+\mu}}, \quad p_2 = \dfrac{q}{p\sqrt{1+\mu}} \\ \alpha_1 = \varepsilon\hat{\alpha}_1, \quad \alpha_2 = \varepsilon\hat{\alpha}_2, \quad F_0 = \varepsilon^2\hat{F}_0 \\ \tilde{\alpha}_1 = p\hat{\alpha}_1, \quad \tilde{\alpha}_2 = \dfrac{1}{2}\hat{\alpha}_2 \end{cases} \qquad (0)$$

and by replacing variables with

$$\begin{cases} \tau = \Omega t, \quad \beta = \dfrac{x}{l} \\ \beta = \varepsilon\hat{\beta}, \quad \varphi = \varepsilon\hat{\varphi}, \quad \tilde{\varphi} = \dfrac{1}{2}\sqrt{\dfrac{\mu}{2(1+\mu)}}\hat{\varphi}, \end{cases} \qquad (0)$$

where ε is a small positive parameter:

$$0 < \varepsilon \le 1. \qquad (0)$$

By considering, also, the relation valid for harmonic functions

$$\tilde{\varphi}'' = p_2^2\tilde{\varphi}, \qquad (0)$$

system (0) becomes:

$$\begin{cases} \hat{\beta}'' + p_1^2\hat{\beta} = \varepsilon\left(\hat{F}_0 p_1^2\cos\tau - 2\tilde{\alpha}_1 p_1^2\hat{\beta}' - \right. \\ \quad \left. - 8p_2^2\tilde{\varphi}^2 + 8\tilde{\varphi}'^2\right) + O\left(\varepsilon^2\right) \\ \tilde{\varphi}'' + p_2^2\tilde{\varphi} = \varepsilon\left(-4\tilde{\alpha}_2 p_2\tilde{\varphi}' - p_1^2\tilde{\varphi}\hat{\beta}\right) + O\left(\varepsilon^2\right). \end{cases} \qquad (0)$$

According to the purpose of the paper, values $p_1=1$ and $p_2=1/2$ will be assumed in the following.

3.STUDY OF THE DYNAMIC SYSTEM

The analysis of the dynamic system will be made in the neighborhood of the equilibrium position, by means of a methodology suggested by Bappaditya Banerjee, Anil K. Bajaj .

The motion equations of a two-degrees-of-freedom system can be written, in general, as

$$\begin{cases} \ddot{\zeta}_1 + C_1\dot{\zeta}_1 + \omega_1^2\zeta_1 + \overline{Q}_1(\zeta_1,\dot{\zeta}_1,\zeta_2,\dot{\zeta}_2) = \\ \quad = \overline{F}_1 \cos t \\ \ddot{\zeta}_2 + C_3\dot{\zeta}_1 + \omega_2^2\zeta_1 + \overline{Q}_2(\zeta_1,\dot{\zeta}_1,\zeta_2,\dot{\zeta}_2) = \\ \quad = \overline{F}_2 \cos t, \end{cases} \qquad (0)$$

where the coefficients have the known significance:

C_1, C_2 – damping coefficients;

ω_1, ω_2 – circular eigenfrequencies of the linear system;

$\overline{F}_1 \cos t$, $\overline{F}_2 \cos t$ – harmonic perturbations;

$\overline{Q}_1$, $\overline{Q}_2$ – second order non-linear effects, depending on the coordinates and on the velocities.

By choosing a small parameter ε, as well as a linear damping force of the same order in ε and by using the transformations

$$\begin{cases} C_1 = \varepsilon C_2, \quad C_3 = \varepsilon C_4 \\ \overline{Q}_1 = \varepsilon Q_1, \quad \overline{Q}_2 = \varepsilon Q_2 \\ \overline{F}_1 = \varepsilon F_1, \quad \overline{F}_2 = \varepsilon F_2 \\ \omega_1^2 = 1 + 2\varepsilon\sigma_1, \quad \omega_2^2 = \frac{1}{4} + \varepsilon\sigma_2, \end{cases} \qquad (0)$$

equations (0) can be rewritten as

$$\begin{cases} \ddot{\zeta}_1 + \dot{\zeta}_1 = \\ \quad = \varepsilon\left(-Q_1 + F_1 \cos t - 2\sigma_1\zeta_1 - C_2\dot{\zeta}_1\right) \\ \ddot{\zeta}_2 + \frac{1}{4}\dot{\zeta}_2 = \\ \quad = \varepsilon\left(-Q_2 + F_2 \cos t - \sigma_2\zeta_2 - C_4\dot{\zeta}_2\right), \end{cases} \qquad (0)$$

where

$$\begin{cases} Q_1(\zeta_1,\dot{\zeta}_1,\zeta_2,\dot{\zeta}_2) = a_{11}\zeta_1^2 + a_{13}\zeta_1\zeta_2 + \\ \quad + a_{22}\dot{\zeta}_1^2 + a_{24}\dot{\zeta}_1\dot{\zeta}_2 + a_{33}\zeta_2^2 + a_{44}\dot{\zeta}_2^2 \\ Q_2(\zeta_1,\dot{\zeta}_1,\zeta_2,\dot{\zeta}_2) = b_{11}\zeta_1^2 + b_{13}\zeta_1\zeta_2 + \\ \quad + b_{22}\dot{\zeta}_1^2 + b_{24}\dot{\zeta}_1\dot{\zeta}_2 + b_{33}\zeta_2^2 + b_{44}\dot{\zeta}_2^2. \end{cases} \qquad (0)$$

According to the above mentioned authors, by defining the state vector

$$\{Z\} = \begin{Bmatrix} \hat{\beta} \\ \hat{\beta}' \\ \tilde{\varphi} \\ \tilde{\varphi}' \end{Bmatrix} = \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix}, \qquad (0)$$

system (0) can be rewritten as

$$\{Z\}' = [A]\{Z\} + \varepsilon\{h_1\}, \qquad (0)$$

where

$$\{h_1\} = \begin{Bmatrix} 0 \\ \hat{F}_0 \cos\tau - 2\tilde{\alpha}_1 z_2 - 2z_3^2 + 8z_4^2 \\ 0 \\ -z_1 z_3 - 2\tilde{\alpha}_2 z_4 \end{Bmatrix}. \qquad (0)$$

By comparing systems (0) and (0) and by taking into account relations (0), the following coefficients are obtained:

$$a_{33} = 2, \quad a_{44} = -8, \quad b_{13} = 1, \quad b_{24} = 0. \qquad (0)$$

With notations

$$F_1 = \alpha\overline{F}_1, \quad C_2 = 2\overline{\xi}_1, \quad C_4 = 2\overline{\xi}_2 \qquad (0)$$

and

$$B = \frac{1}{2}\left(b_{13} + \frac{b_{24}}{2}\right) = \frac{1}{2}, \quad \alpha = \frac{1}{B} = 2, \qquad (0)$$

the expressions of the modal amplitudes a_1 and a_2, respectively, can be found:

– for a single vibration mode (a_2=0),

$$a_1 = \frac{\overline{F}}{2\sqrt{\overline{\xi}_1^2 + \sigma_1^2}}; \qquad (0)$$

– for coupled vibration modes,

$$\begin{cases} a_1 = 2\sqrt{\overline{\xi}_2^2 + \sigma_2^2} \\ a_2^4 + 4\left(\overline{\xi}_1\overline{\xi}_2 - \sigma_1\sigma_2\right)a_2^2 + \\ \quad + 4\left(\overline{\xi}_1^2 + \sigma_1^2\right)\left(\overline{\xi}_2^2 + \sigma_2^2\right) - \frac{\overline{F}_2^2}{4} = 0. \end{cases} \qquad (0)$$

From relations (0) and (0) it follows that the evolution of the dynamic system can be analyzed by means of Hopf bifurcation. It is known that, in this case, periodic solutions can develop in two ways :

– soft, when a stable equilibrium position, for $\lambda<\lambda_0$, becomes unstable, for $\lambda>\lambda_0$, generating a stable periodical solution; the orbit develops from a point in the phase plane, to closed curves, through continuous deformations, for $\lambda>\lambda_0$;

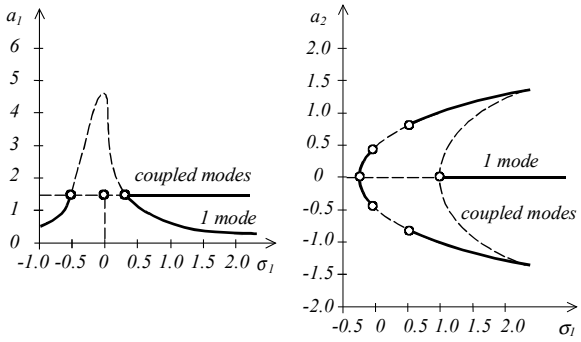
– hard, when a stable equilibrium position, for $\lambda<\lambda_0$, becomes unstable, for $\lambda>\lambda_0$, stable periodical solution being possible; when λ_0 value is slightly exceeded, the periodical motion is generated through a jump.

Staring from the theoretical considerations above, the response curves of the system will be plotted and the results will be interpreted.

4.RESULTS AND CONCLUSIONS

For the values $\overline{\xi}_1 = \overline{\xi}_2 = 0.10$, $\sigma_2 = 0.66$, $\overline{F} = 1.0$, the curves in Figures 3 a and 3 b have been plotted. The continuous line represents the

stable solution, while the broken line the unstable one. Hopf bifurcation points are figured in each case.



a) b)

Fig. 3. Graphs of the modal amplitudes

From the qualitative analysis of the numerical and graphical results, it can be noticed that:

- for $\bar{\xi}_1 = \bar{\xi}_2 > 0.214$ no Hopf bifurcation appears;
- coupled modes lead to limit cycle type motions;
- for σ_1 values outside interval $(-0.3, -0.15)$, a doubling of the bifurcation occurs, which expresses the possibility of evolution towards chaos ;
- a hard generation can be remarked.

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