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ABOUT THE VIBRATIONS OF THE MECHANICAL SYSTEMS WITH NON-LINEAR DAMPING

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ABSTRACT

This work will analyze the non-linear vibrations of the mechanical systems, actually the ones with non-linear damping. If we consider damping $F(x,\dot{x}) = -\dot{x}(k_1 - k_2\dot{x}^2); k_1,k_2 \in R$ *then the differential equation that characterizes the movement of the system is a Rayleigh one. Using a derivation and a substitution, the differential equation becomes a Van der Pol one, for which we find the analytical approximate solution.*

KEYWORDS: vibration, mechanical system, non-linear

1. Introduction

The free vibrations of the elastic systems with non-linear damping [1] are characterized by the next differential equation:

$$m\ddot{x} + F(x,\dot{x}) + kx = 0, \tag{1}$$

where m is the mass of that kind of system, $F(x,\dot{x})$ is the damping force, x is the movement and k is the elastic coefficient.

If

$$F(x,\dot{x}) = -\dot{x}(k_1 - k_2\dot{x}^2); k_1, k_2 \in R; \tag{2}$$

the equation (1) is a Rayleigh one.

For $\dot{x} < \sqrt{\frac{k_1}{k_2}}$, in the system there appear autovibrations, and for $\dot{x} > \sqrt{\frac{k_1}{k_2}}$ the vibrations are damped.

From (1) and (2) the non-linear differential equation results:

$$m\ddot{x} - (k_1 - k_2\dot{x}^2)\dot{x} + kx = 0, \tag{3}$$

for which we shall determine the analytical aproximate solutions [2].

2. Analytical solution

Using the derivation related to the time from the differential equation (3) it results:

$$m\dddot{x} + 3k_2\ddot{x}\dot{x}^2 - k_1\ddot{x} + k\dot{x} = 0, \tag{4}$$

and for

$$\dot{x} = y, k_1 = c_1, 3k_2 = c_2, \tag{5}$$

we obtain

$$m\ddot{y} - (c_1 - c_2 y^2)\dot{y} + ky = 0, \tag{6}$$

the Van der Pool type equation where we change the variable

$$\tau = \sqrt{\frac{k}{m}} \cdot t \tag{7}$$

and we know that

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{d\tau} \cdot \frac{d\tau}{dt} = \sqrt{\frac{k}{m}} \cdot \frac{dy}{d\tau}, \tag{8}$$

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{d\tau} \cdot \frac{d\tau}{dt} = \frac{k}{m} \cdot \frac{d^2 y}{d\tau^2}. \qquad (9)$$

The differential equation (6) becomes:

$$\frac{d^2 y}{d\tau^2} - \frac{c_1 - c_2}{\sqrt{km}} \cdot y^2 \cdot \frac{dy}{d\tau} + y = 0. \qquad (10)$$

Using the change of function as in:

$$y = \sqrt{\frac{c_1}{c_2}} \cdot z \qquad (11)$$

the equation (10) turns into:

$$\frac{d^2 z}{d\tau^2} - \varepsilon(1 - z^2)\frac{dz}{d\tau} + z = 0 \qquad (12)$$

and

$$\varepsilon = \frac{c_1}{\sqrt{km}}. \qquad (13)$$

We shall solve the differential equation (12) on the conditions of the limit:

$$z(0) = z_0, \frac{dz}{d\tau}(0) = 0, \qquad (14)$$

taking the solution for the first approximation:

$$z = A\sin(\omega\tau + \varphi) \qquad (15)$$

or

$$z = A\sin\varphi_1 \qquad (16)$$

where the amplitude A and the phase φ are depend on τ and varies slowly in time. Because of the slow variation, their derivations $\dot{A} = \dot{A}(t)$, $\dot{\varphi} = \dot{\varphi}(t)$ suffer little modifications in a period of movement so their medium values are equal to their functions.

Because

$$\varphi_1 = \omega t + \varphi \qquad (16')$$

for a period 2π, we shall make the approximations:

$$\frac{dA}{d\tau} = \frac{\varepsilon}{2\pi}\int_0^{2\pi}(1 - A^2\sin^2\varphi_1)(A\cos\varphi_1)\cos\varphi_1 d\varphi_1, \qquad (17)$$

$$\frac{d\varphi}{d\tau} = -\frac{\varepsilon}{2\pi A}\int_0^{2\pi}(1 - A^2\sin^2\varphi_1)(A\cos\varphi_1)\sin\varphi_1 d\varphi_1. \qquad (18)$$

The differential equation (17) leads to

$$\begin{aligned}
\frac{dA}{d\tau} &= \frac{\varepsilon}{2\pi}\left[\int_0^{2\pi} A\cos^2\varphi_1 d\varphi_1 - \right. \\
&\left. - A^3\int_0^{2\pi}\sin^2\varphi_1\cos^2\varphi_1 d\varphi_1\right] = \\
&= \frac{\varepsilon}{2\pi}\left[A\int_0^{2\pi}\frac{1}{2}(1 + \cos 2\varphi_1)d\varphi_1 - \right. \\
&\left. -\left(\frac{A^3}{4}\int_0^{2\pi}\sin^2 2\varphi_1 d\varphi_1\right]\right. = \\
&= \frac{\varepsilon}{2\pi}\left[\frac{A}{2}\left(\varphi_1 + \frac{1}{2}\sin 2\varphi_1\right)\Bigg|_0^{2\pi} - \right. \\
&\left. - \frac{A^3}{4}\int_0^{2\pi}\frac{1}{2}(1 - \cos 4\varphi_1)d\varphi_1\right] = \\
&= \frac{\varepsilon}{2\pi}\left[A\pi - \frac{A^3}{8}\left(\varphi_1 - \frac{1}{4}\sin 4\varphi_1\right)\Bigg|_0^{2\pi}\right] = \\
&= \frac{1}{8}\varepsilon A(4 - A^2).
\end{aligned} \qquad (19)$$

The differential equation

$$\frac{dA}{d\tau} = \frac{1}{8}\varepsilon A(4 - A^2) \qquad (20)$$

has separable variable

$$\frac{dA}{A(4 - A^2)} = \frac{1}{8}\varepsilon d\tau, \qquad (21)$$

or

$$\int\frac{dA}{A(4 - A^2)} = \frac{1}{8}\varepsilon\int d\tau \qquad (22)$$

and then

$$\frac{1}{4}\int\frac{dA}{A} + \frac{1}{8}\int\frac{dA}{2 - A} - \frac{1}{8}\int\frac{dA}{2 + A} = \frac{1}{8}\varepsilon\tau. \qquad (23)$$

By solving the integers it results:

$$\frac{1}{8}ln\frac{A^2}{4-A^2}=\frac{1}{8}\varepsilon\tau+\frac{1}{8}ln\,c, \quad c\in R, c>0 \tag{24}$$

and then

$$\frac{A^2}{4-A^2}=c\cdot e^{\varepsilon\tau}. \tag{25}$$

From the differential equation (18) we obtain:

$$\frac{d\varphi}{d\tau}=-\frac{\varepsilon}{2\pi A}\left[\int_0^{2\pi}a\sin\varphi_1\cos\varphi_1 d\varphi_1-A^3\int_0^{2\pi}\sin^3\varphi_1\cos\varphi_1 d\varphi_1\right]=$$
$$=-\frac{\varepsilon}{2\pi A}\left[A\int_0^{2\pi}\frac{1}{2}\sin 2\varphi_1 d\varphi_1-A^3\int_0^{2\pi}\sin^3\varphi_1 d(\sin\varphi_1)\right]=$$
$$=-\frac{\varepsilon}{2\pi A}\left[\frac{a}{2}\cdot\frac{1}{2}(-\cos 2\varphi_1)\Big|_0^{2\pi}-\frac{A^3}{4}\sin^4\varphi_1\Big|_0^{2\pi}\right]=0, \tag{26}$$

so $\varphi=\varphi_0=const.$

The c,φ_0 constants of integration are obtained from the initial condition(14)

$$z(0)=A(0)\sin\varphi_0=z_0; \tag{27}$$

$$\frac{dz}{d\tau}(0)=A(0)\varphi_0\cos\varphi_0=0, \tag{28}$$

so, it results

$$\varphi_0=\frac{\pi}{2}, A(0)=z_0 \tag{29}$$

and

$$c=\frac{z_0^2}{4-z_0^2}. \tag{30}$$

Back to (25) with these solutions, we get:

$$\frac{A^2}{4-A^2}=\frac{z_0^2}{4-z_0^2}e^{\varepsilon\tau}, \tag{31}$$

so

$$A=2z_0\sqrt{\frac{e^{\varepsilon\tau}}{4+z_0^2(e^{\varepsilon\tau}-1)}}, \tag{32}$$

or

$$A(\tau)=\frac{2z_0e^{\frac{\varepsilon\tau}{2}}}{\sqrt{4+z_0^2(e^{\varepsilon\tau}-1)}} \tag{33}$$

and the approximate solution of the differential equation (12), depending on (15) is:

$$z(\tau)=\frac{2z_0e^{\frac{\varepsilon\tau}{2}}}{\sqrt{4+z_0^2(e^{\varepsilon\tau}-1)}}\sin(\omega\tau+\frac{\pi}{2}), \tag{34}$$

or

$$z(\tau)=\frac{2z_0e^{\frac{\varepsilon\tau}{2}}}{\sqrt{4+z_0^2(e^{\varepsilon\tau}-1)}}\cos\omega\tau, \tag{35}$$

is a solution which describes an oscillating periodic move.

From (7), (13), (11), (5) and (35) it results

$$z(t)=\frac{2z_0e^{\frac{1}{2}\cdot\frac{k_1}{m}\cdot t}}{\sqrt{4+z_0^2(e^{\frac{k_1}{m}\cdot t}-1)}}\cos\left(\omega\sqrt{\frac{k}{m}}\cdot t\right), \tag{36}$$

$$y(t)=2\sqrt{\frac{k_1}{3k_2}}\cdot z_0\cdot\frac{e^{\frac{1}{2}\frac{k_1}{m}\cdot t}}{\sqrt{4+z_0^2(e^{\frac{k_1}{m}t}-1)}}\cos\omega\sqrt{\frac{k}{m}}\cdot t \tag{37}$$

and then the expression in time becomes

$$x(t)=2z_0\sqrt{\frac{k_1}{3k_2}}\int_0^{2\pi}\frac{e^{\frac{1}{2}\frac{k_1}{m}\cdot t}}{\sqrt{4+z_0^2(e^{\frac{k_1}{m}t}-1)}}\cos\omega\sqrt{\frac{k}{m}}\cdot t dt, \tag{38}$$

where

$$z_0=\sqrt{\frac{3k_2}{k_1}}\cdot\dot{x}(0). \tag{39}$$

Particular case.

For $z_0 = 2$, the differential equation (3) has the approximate analitical solution:

$$x(t) = 2\sqrt{\frac{k_1}{3k_2}}\int_0^{2\pi}\cos\left(\omega\sqrt{\frac{k}{m}}\cdot t\right)dt =$$
$$= \frac{1}{\omega}\sqrt{\frac{m}{k}}\cdot 2\sqrt{\frac{k_1}{3k_2}}\cdot\sin\omega\sqrt{\frac{k}{m}}\cdot t\Big|_0^{2\pi} = \quad (40)$$
$$= \frac{2}{\omega}\sqrt{\frac{k_1 m}{3k_2 k}}\cdot\sin 2\pi\omega\sqrt{\frac{k}{m}}.$$

3. Numerical solution

For a set of particular data the graph of the analytical approximate solution of the differential equation (3), as said in (40), is presented in fig 1.

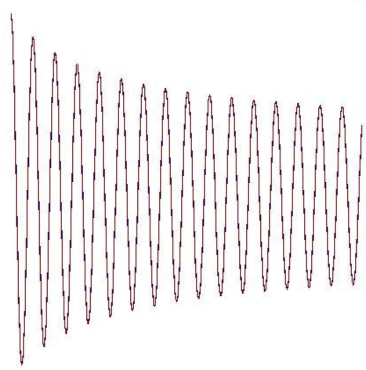

Figure 1. The graph of the solution after (40)

For the same set of data, the differential equation (3) was solved numerically using the fourth order of the Runge-Kutta method [3] and the solution in Figure 2 has been obtained.

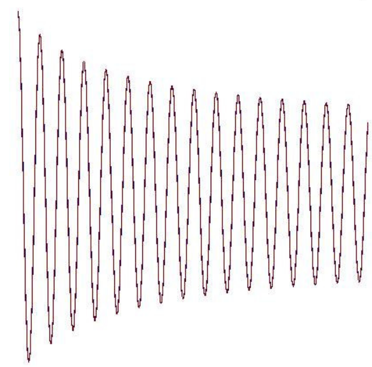

Figure 2. The graph of the solution using the Runge-Kutta method.

Both graphs were obtained in 100 seconds and the biggest difference of data for the same moment was 0.2 x 10^{-4}.

4. Conclusion

The Rayleigh differential equation that describe the non-linear vibrations of a mechanical system can be solved as a Van der Pol equation in order to find a very precise analytical approximate solution.

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