THE ANNALS OF “DUNAREA DE JOS” UNIVERSITY OF GALATI
FASCICLE XIV MECHANICAL ENGINEERING, ISSN 1224-5615
2009

PERIODIC SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS THAT SHOW THE NON-LINEAR VIBRATIONS OF THE MECHANICAL SYSTEMS

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ABSTRACT

Many phenomenons of mechanical nature possess non-linear vibrations, their mathematical forming operation leading to differential equations or to systems of differential non-linear equations.

In this work it is shown that we can determine aproximate analitical solutions for non-linear differential equations, such as $\ddot{x}+\varepsilon f\left(x,\dot{x}\right)+x=F\left(t\right)$. *We use the perturbations method for homogeneous and non-homogeneous for low parameters and we show that in special situations this equations are Van der Pol or Duffing equations.*

KEYWORDS: vibration, mechanical system, non-linear

1. Introduction

Generally, phenomenons of mechanical nature possess non-linear vibrations and they are described [1] by differential equations such as:

$$\ddot{x}+f\left(x,\dot{x},t\right)=0 \qquad (1)$$

An important category of non-linear equations characterizes the elastic systems with small non-linear, distinguished by the presence of a parameter in the equation.

We shall search for aproximate analitical solutions for the differential equations such as:

$$\ddot{x}+\varepsilon f\left(x,\dot{x}\right)+x=F\left(t\right), \qquad (2)$$

where x is the movement, $\dot{x}$ - the speed, ε - small parameter and F is the excitation force, using the perturbations method.

In special situations, this kind of equations are Van der Pol or Duffing differential equations.

2. Free oscillations

To illustrate the procedure we consider the differential equation:

$$\ddot{x}-\varepsilon\left[a\dot{x}\left(1-x^2\right)-bx^3\right]+x=0, \qquad (3)$$

where ε is supposed to be small. Assuming that the ε-term can be neglected, (3) reduces to:

$$\ddot{x}+x=0, \qquad (4)$$

the solution of which is:

$$x=\alpha\cos t+\beta\sin t, \qquad (5)$$

and this solution is periodic with 2π period.

If the periodic solutions of (3) exist, then the effect of the ε-term is to change the period slightly to $2\pi/\omega$, say, where $\omega=\omega\left(\varepsilon\right)$ differs slightly from unity [2].

It is supposed then that

$$\omega = \omega(\varepsilon) = 1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \ldots \qquad (6)$$

Change the independent variable in (3) by means of the substitution:

$$\theta = \omega t, \qquad (6')$$

to give

$$\omega^2 x'' - \varepsilon\left[a\omega x'\left(1-x^2\right) - bx^3\right] + x = 0, \qquad (7)$$

where primes denote differentiation with respect to θ.

As usual, let

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots . \qquad (8)$$

Since only one additional condition can be imposed on *x* other than periodicity, it is supposed that

$$x(0)=0, \qquad (8')$$

which implies that

$$x_0(0) = x_1(0) = \ldots = 0. \qquad (9)$$

Substituting the series for ω^2, $\omega x'\left(1-x^2\right)$, x^3, x in (7) and equating the coefficients of the powers of ε to zero, we deduce from the vanishing of the coefficient of ε^0 that

$$x_0'' + x_0 = 0, \qquad (10)$$

whence

$$x_0 = \alpha \sin\theta \qquad (11)$$

since $x_0(0)=0$; here α is a constant to be determined from the condition that the solution is to be periodic.

Similarly, equating to zero the coefficient of ε, it is found that

$$\begin{aligned} x_1'' + x_1 &= a\alpha\left(1-\frac{\alpha^2}{4}\right)\cos\theta + \\ &+\alpha\left(2\omega_1 - \frac{3b\alpha^2}{4}\right)\sin\theta + \\ &+\frac{a\alpha^3}{4}\cos 3\theta + \frac{b\alpha^3}{4}\sin 3\theta . \end{aligned} \qquad (12)$$

If the first two terms on the right-hand side of (12) are allowed to persist, their contribution to the particular integral is nonperiodic.

Hence α and ω_1 are chosen such that these two terms vanish, that is

$$1-\frac{\alpha^2}{4} = 0 \text{ , i.e. } \alpha = 2, \qquad (13)$$

and

$$2\omega_1 - \frac{3b\alpha^2}{4} = 0 \text{ , i.e. } \omega_1 = \frac{3b}{2} . \qquad (14)$$

With these values of α and ω_1 the solution of (12) is now

$$x_1 = \beta\sin\theta + \frac{a\left(\cos\theta - \cos 3\theta\right)}{4} - \frac{b\sin 3\theta}{4}, \qquad (15)$$

since $x_1(0) = 0$. The constant β is determined later using once again the condition of periodicity.

Since α=2, (11) has the solution

$$x_0 = 2\sin\theta . \qquad (16)$$

From the vanishing of the coefficient of ε^2, we find, after some reduction, that

$$\begin{aligned} x_2'' + x_2 &= \left(\frac{b}{2} - \frac{3ab}{2} - 2\beta\right)\cos\theta - \\ &-\left(6\beta b - \frac{15b^2}{4} - \frac{5a^2}{4} + a - 4\omega^2\right)\sin\theta + \\ &+\left(a\beta + 2\beta - \frac{3ab}{4}\right)\cos 3\theta + \\ &+\left(3\beta b - \frac{21b^2}{4} - a^2 - \frac{a}{2}\right)\sin 3\theta - \\ &-\left(\frac{3ab}{2} + \frac{b}{2}\right)\cos 5\theta + \\ &+\left(\frac{3a^2}{4} - \frac{3b^2}{4} + \frac{a}{4}\right)\sin 5\theta . \end{aligned} \qquad (17)$$

Again, if the solution is to be periodic, the coefficients of the $\cos\theta, \sin\theta$ on the right-hand side of (17) should be zero.

This means that

$$\beta = \frac{b(1-3a)}{4};$$
$$\omega_2 = -\frac{9b^2(1+2a)+a(5a-4)}{16} \qquad (18)$$

on substituting for β; making the substitution for β in (15)

$$x_1 = \frac{b(1-3a)}{4}\sin\theta + \\ +\frac{a}{4}(\cos\theta - \cos 3\theta) - \frac{b}{4}\sin 3\theta. \qquad (19)$$

Proceeding this way, (17) we can solve, the constant of integration, an ω_3 being evaluated from the condition that x_3 should be periodic.

The solution obtained so far has the form

$$x = 2\sin\omega t + \frac{\varepsilon}{4}\cdot[a(\cos\omega t - \cos 3\omega t) + \\ +b(1-3a)\sin\omega t - b\sin 3\omega t] + \ldots, \qquad (20)$$

where

$$\omega = 1 + \frac{3b}{4}\varepsilon - \frac{9b^2(1+2a)+a(5a-4)}{16}\varepsilon^2 \ldots. \qquad (21)$$

Particular cases:

1. If $a=1, b=0$, then, (3) is Van der Pol's differential equation. The solution which has $2\pi/\omega$ period and such that x(0)=0, is

$$x = 2\sin\omega t + \frac{\varepsilon}{4}(\cos\omega t - \cos 3\omega t)\cdots, \qquad (22)$$

with

$$\omega = 1 - \frac{\varepsilon^2}{16}. \qquad (23)$$

2. If $a=0, b=1$, then (3) has a solution which has x(0)=0 and $2\pi/\omega$ period:

$$x = 2\sin\omega t + \frac{\varepsilon}{4}(\sin\omega t - \sin 3\omega t)\cdots, \qquad (24)$$

with

$$\omega = 1 + \frac{3\varepsilon}{2} - \frac{9\varepsilon^2}{16}. \qquad (25)$$

3. Forced oscillations.

We consider the differential equation

$$\ddot{x} - \varepsilon\cdot\left[a\dot{x}(1-x^2) - bx^3\right] + x = \cos\omega t \qquad (26)$$

and try to find a solution which has $2\pi/\omega$ period and which satisfies

$$x(0)=\alpha. \qquad (26')$$

Again we put [3]

$$\theta = \omega t, \qquad (27)$$

thus modifying the above equation to

$$\omega^2 x'' - \varepsilon\cdot\left[a\omega x'(1-x^2) - bx^3\right] + x = \cos\theta, \qquad (28)$$

where primes denote differentiation with respect to θ. If the solution of (26) has period $2\pi/\omega$, then the solution of (28) has period 2π.

In this case we take $\omega(\varepsilon)$ in the form of:

$$\omega(\varepsilon) = \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \ldots \qquad (29)$$

an as usual equation as the following:

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + \ldots. \qquad (30)$$

The initial condition requires that

$$x_0(0) = \alpha; \quad x_1(0) = x_2(0) = \ldots = 0. \qquad (31)$$

Substituting the appropriate series in (28) and then equating the coefficients of the corresponding powers of ε on each side of the resulting equation, we find from the coefficient of ε^0 that

$$\omega_0^2 x_0'' + x_0 = \cos\theta. \qquad (32)$$

The solution of this differential equation is

$$x_0 = \alpha\cos\frac{\theta}{\omega_0} + \beta\sin\frac{\theta}{\omega_0} + \frac{1}{1-\omega_0^2}\cos\theta, \qquad (33)$$

from which it is evident that $\omega_0 \neq 1$ for a periodic solution.

Hence, if the solution of (33) is to have periodicity 2π, it follows that α=β=0 and

$$x_0 = \frac{1}{1-\omega_0^2}\cos\theta \cdot \tag{34}$$

The initial condition $x_0(0)=\alpha$ requires that

$$\alpha = \frac{1}{1-\omega_0^2}, \text{ i.e. } \omega_0^2 = 1-\frac{1}{\alpha}, \tag{35}$$

so that

$$x_0 = \alpha\cos\theta \cdot \tag{36}$$

If 0 < α < 1 then no periodic solution exists .

Equating the coefficients of ε in the expansion of (28), we have:

$$\omega_0^2 x_1'' + x_1 = -2\omega_0\omega_1 x_0'' - bx_0^3 - a\omega_0 x_0'\left(1-x_0^2\right) \tag{37}$$

which, on using (36) becomes:

$$\omega_0^2 x_1'' + x_1 = \alpha\left(2\omega_0\omega_1 - \frac{3b\alpha^2}{4}\right)\cos\theta - \\ -\alpha a\omega_0\left(1-\frac{\alpha^2}{4}\right)\sin\theta - \\ -\frac{b\alpha^3}{4}\cos 3\theta - \frac{a\omega_0\alpha^3}{4}\sin 3\theta \cdot \tag{38}$$

Once more, the only solution having period 2π arises from the particular integral, and is

$$x_1 = \alpha^2\left(2\omega_0\omega_1 - \frac{3b\alpha^2}{4}\right)\cos\theta - \\ -a\omega_0\alpha^2\left(1-\frac{\alpha^2}{4}\right)\sin\theta - \\ -\frac{b\alpha^4}{4(9-8\alpha)}\cos 3\theta - \frac{a\omega_0\alpha^4}{4(9-8\alpha)}\sin 3\theta \cdot \tag{39}$$

In order to satisfy the condition $x_1(0)=0$, it is evident that:

$$2\omega_0\omega_1 = \frac{3b\alpha^2}{4} + \frac{b\alpha^4}{4(9-8\alpha)} \cdot \tag{40}$$

Whence, on substituting for $2\omega_0\omega_1$ in the previous expression for x_1, we see that:

$$x_1 = \frac{b\alpha^4}{4(9-8\alpha)}(\cos\theta - \cos 3\theta) - \\ -a\omega_0\alpha^2\left[\left(1-\frac{\alpha^2}{4}\right)\sin\theta + \frac{\alpha^2}{4(9-8\alpha)}\sin 3\theta\right] . \tag{41}$$

This process is continued.

Particular cases:

1. If $a=1, b=0$ then we have Van der Pol's equation with a forcing term.

The solution which has period $2\pi/\omega$ in *t* is

$$x = \alpha\cos\omega t - \\ -\varepsilon\alpha^2\sqrt{1-\frac{1}{\alpha}}\left[\left(1-\frac{\alpha^2}{4}\right)\sin\omega t + \frac{\alpha^2}{4(9-8\alpha)}\sin 3\omega t\right] \tag{42}$$

since from (35)

$$\omega_0 = \sqrt{1-\frac{1}{\alpha}} \cdot \tag{43}$$

2. If $a=0, b=1$, then the equation is Duffing's equation.

The solution which has period $2\pi/\omega$ is in this case

$$x = \alpha\cos\omega t + \\ +\frac{\varepsilon\alpha^4}{4\left(9-8\alpha^2\right)}(\cos\omega t - \cos 3\omega t)\ldots . \tag{44}$$

4. Conclusion

The non-linear equations (2) arises in a number of physical applications and includes the special cases known as Van der Pol's and Duffing's equations.

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