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THE UTILIZATION OF 3X3 MATRIXES IN SETTING KINEMATICS CHAINS ANALYSIS FOR ROBOTS COMPONENTS

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ABSTRACT

Within the kinematic study of a rigid solid, its motion is studied face-to-face by a reference system, fixing positions, speeds and accelerations of a reference system, solidarily connected to the rigid solid.

KEYWORDS: kinematics, analysis, chains, transformation matrix.

1. Introduction

The author considers, in this paper, the rotation of the rigid solid around a certain axis (Δ), with an angle φ (Figure 1). The motion of a solid rigid is studied face-to-face by a reference system with fixing position.

2. Description method of analysis

We deem the rotation axis (Δ), defined face-to-face by a tri-orthogonal system $O_0x_0y_0z_0$, which is fixedly connected with the direction cosines

$$\alpha_{11} = \cos\alpha, \alpha_{12} = \cos\beta, \alpha_{13} = \cos\gamma; \qquad (1)$$

and the reference system $O_1x_1y_1z_1$, whose axis O_1x_1 coincides with the rotation axis (Δ), and the axis O_1z_1 is situated in $O_0x_0z_0$ plan.

The positions of the reference system axis $O_1x_1y_1z_1$ are defined in connection with a system fixedly connected by

$$\begin{aligned} O_1x_1 &: \alpha_{11}^1 = \cos\alpha_1, \alpha_{12}^1 = \cos\beta_1, \alpha_{13}^1 = \cos\gamma_1 \\ O_1y_1 &: \alpha_{21}^1 = \cos\alpha_2; \alpha_{22}^1 = \cos\beta_2, \alpha_{23}^1 = \cos\gamma_2 \\ O_1z_1 &: \alpha_{31}^1 = \cos\alpha_3, \alpha_{32}^1 = \cos\beta_3, \alpha_{33}^1 = \cos\gamma_3 \end{aligned} \qquad (2)$$

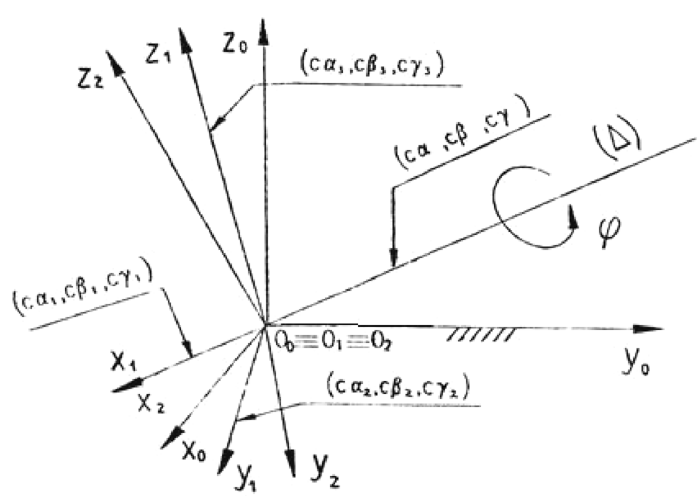

Figure 1. Rotation of the rigid solid around a certain axis (Δ), with an angle φ

The transformation matrix from the reference system $O_1x_1y_1z_1$ to the fixedly reference system is:

$$[a_{10}]^T = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} cos\,\alpha_1 & cos\,\alpha_2 & cos\,\alpha_3 \\ cos\,\beta_1 & cos\,\beta_2 & cos\,\beta_3 \\ cos\,\gamma_1 & cos\,\gamma_2 & cos\,\gamma_3 \end{bmatrix}, \quad (3)$$

respectively from the fixedly reference system $O_0x_0y_0z_0$ to the reference system $O_1x_1y_1z_1$:

$$[a_{10}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} cos\,\alpha_1 & cos\,\beta_1 & cos\,\gamma_1 \\ cos\,\alpha_2 & cos\,\beta_2 & cos\,\gamma_2 \\ cos\,\alpha_3 & cos\,\beta_3 & cos\,\gamma_3 \end{bmatrix}, \quad (4)$$

Using the relations that express the perpendicularity conditions of two axes, we can write

$$cos\,\alpha_1\,cos\,\beta_1 + cos\,\alpha_2\,cos\,\beta_2 + cos\,\alpha_3\,cos\,\beta_3 = 0 \quad (5)$$

$$cos\,\beta_1\,cos\,\gamma_1 + cos\,\beta_2\,cos\,\gamma_1 + cos\,\beta_3\,cos\,\gamma_3 = 0$$

in which, replacing

$$cos\,\alpha_1 = cos\,\alpha; cos\,\beta_1 = cos\,\beta; \quad cos\,\gamma_1 = cos\,\gamma; cos\,\beta_3 = 0; cos\,\beta_2 = sin\,\beta \quad (6)$$

results:

$$cos\,\alpha\,cos\,\beta + cos\,\alpha_2\,sin\,\beta = 0 \Rightarrow cos\,\alpha_2 = -\frac{cos\,\alpha\,cos\,\beta}{sin\,\beta}. \quad (7)$$

$$cos\,\beta\,cos\,\gamma + sin\,\beta\,cos\,\gamma_2 = 0 \Rightarrow cos\,\gamma_2 = -\frac{cos\,\beta\,cos\,\gamma}{sin\,\beta}$$

Having in view the fact that each element of the transformation matrix can be equal to its cofactor, results:

$$cos\,\alpha_3 = cos\,\beta_1\,cos\,\gamma_2 - cos\,\gamma_1\,cos\,\beta_2 = -\frac{cos^2\,\beta\,cos\,\gamma}{sin\,\beta} - cos\,\gamma\,sin\,\beta = -\frac{cos\,\gamma}{sin\,\beta} \quad (8)$$

$$cos\,\gamma_3 = cos\,\alpha_1\,cos\,\beta_2 - cos\,\beta_1\,cos\,\alpha_2 = cos\,\alpha\,sin\,\beta + \frac{cos\,\alpha\,cos^2\,\beta}{sin\,\beta} = \frac{cos\,\alpha}{sin\,\beta}$$

We deem the rotation of a φ angle system $O_1x_1y_1z_1$ around the $O_1x_1 \equiv (\Delta)$ axis. In this case, the transformation matrix becomes

$$\left[R_{0x}^{\varphi}\right] = [a_{21}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\,\varphi & -sin\,\varphi \\ 0 & sin\,\varphi & cos\,\varphi \end{bmatrix}. \quad (9)$$

Considering a point M' defined in $O_2x_2y_2z_2$ system, by vector $\left(r_{M'}^2\right)$, it results from the rotation of φ around its axis (Δ), from the point M belonging to the $O_1x_1y_1z_1$ system, by vector $\left(r_M^1\right)$.

Knowing that the analytic expression of the position vector is invariable face-to-face by rotation of the coordinating axis, we can write:

$$\left(r_{M'}^2\right) = \left(r_M^1\right) \quad (10)$$

whence the result of left multiplying by $[a_{21}]^T$, is:

$$[a_{21}]^T\left(r_{M'}^2\right) = [a_{21}]^T\left(r_M^1\right). \quad (11)$$

We go on writing that

$$\left(r_{M'}^0\right) = [a_{10}]^T\left(r_{M'}^1\right) \quad (12)$$

$$\left(r_M^0\right) = [a_{10}]^T\left(r_M^1\right) \quad (13)$$

which by $[a_{10}]$ left multiplying there results:

$$\left(r_{M'}^1\right) = [a_{10}]\left(r_{M'}^0\right), \quad (14)$$

$$\left(r_M^1\right)=\left[a_{10}\right]\left(r_M^0\right) \tag{15}$$

If we replace (14) and (15) in (11), the result is shown in:

$$\left[a_{21}\right]^T\left(r_{M'}^2\right)=\left[a_{21}\right]^T\left[a_{10}\right]\left(r_M^0\right) \tag{16}$$

whence by $\left[a_{10}\right]^T$ left-multiplying, results:

$$\left[a_{10}\right]^T\left[a_{21}\right]^T\left(r_{M'}^2\right)=\left[a_{10}\right]^T\left[a_{21}\right]^T\left[a_{10}\right]\left(r_M^0\right). \tag{17}$$

Seeing that

$$\left(r_{M'}^0\right)=\left[a_{10}\right]^T\left[a_{21}\right]^T\left(r_{M'}^2\right), \tag{18}$$

the relation (17) becomes

$$\left(r_{M'}^0\right)=\left[a_{10}\right]^T\left[a_{21}\right]^T\left[a_{10}\right]\left(r_M^0\right) \tag{19}$$

and the transformation matrix is

$$\left[a_r\right]^T=\left[a_{10}\right]^T\left[a_{21}\right]^T\left[a_{10}\right]. \tag{20}$$

The matrix $\left[a_r\right]$, defined by the relation (20), define the rotation of the rigid solid around the axis (Δ), with an angle φ, face-to-face by the reference system fixedly connected.

Calculating the matrix products

$$\left[a_{10}\right]^T\left[a_{21}\right]^T=$$
$$=\begin{bmatrix} c\alpha & -c\alpha\dfrac{c\beta}{s\beta} & -\dfrac{c\gamma}{s\beta} \\ c\beta & s\beta & 0 \\ c\gamma & -c\beta\dfrac{c\gamma}{s\beta} & \dfrac{c\alpha}{s\beta} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix}= \tag{21}$$
$$=\left[a_{20}\right]^T$$

and

$$\left[a_{20}\right]^T\left[a_{10}\right]=$$
$$=\begin{bmatrix} c\alpha & -\dfrac{c\alpha c\beta c\varphi+c\gamma s\varphi}{s\beta} & \dfrac{-c\gamma c\varphi+c\alpha c\beta s\varphi}{s\beta} \\ c\beta & s\beta c\varphi & -s\beta s\varphi \\ c\gamma & \dfrac{c\alpha s\varphi-c\beta c\gamma c\varphi}{s\beta} & \dfrac{c\beta c\gamma s\varphi+c\alpha c\varphi}{s\beta} \end{bmatrix}x$$
$$x\begin{bmatrix} c\alpha & c\beta & c\gamma \\ -\dfrac{c\alpha c\beta}{s\beta} & s\beta & -\dfrac{c\beta c\gamma}{s\beta} \\ -\dfrac{c\varphi}{s\beta} & 0 & \dfrac{c\alpha}{s\beta} \end{bmatrix} \tag{22}$$

finally, the transformation matrix the relation becomes:

$$\left[a_r\right]^T=\left[a_{10}\right]^T\left[a_{21}\right]^T\left[a_{10}\right]=$$
$$=\begin{bmatrix} \alpha_{11}^r & \alpha_{21}^r & \alpha_{31}^r \\ \alpha_{12}^r & \alpha_{22}^r & \alpha_{32}^r \\ \alpha_{13}^r & \alpha_{23}^r & \alpha_{33}^r \end{bmatrix}, \tag{23}$$

where

$$\alpha_{11}^r=c^2\alpha+$$
$$+\frac{c^2\alpha c^2\beta c\gamma+c\alpha c\beta c\gamma s\varphi-c\alpha c\beta c\gamma s\varphi+c^2\gamma c\varphi}{s^2\beta}$$

$$\alpha_{21}^r=c\alpha c\beta-\frac{s\beta c\varphi c\alpha c\beta}{s\beta}+\frac{s\beta c\gamma s\varphi}{s\beta}$$

$$\alpha_{31}^r=c\gamma c\alpha-\frac{c^2\alpha c\beta s\varphi}{s^2\beta}+\frac{c\alpha c^2\beta c\gamma c\varphi}{s^2\beta}-$$
$$-\frac{c\beta c^2\gamma s\varphi+c\alpha c\gamma c\varphi}{s^2\beta}$$

$$\alpha_{12}^r=c\alpha c\beta-\frac{c\alpha c\beta s\beta c\varphi+c\gamma s\beta s\varphi}{s\beta}$$

$$\alpha_{22}^r=c^2\beta+s^2\beta c\varphi$$

$$\alpha_{32}^r=c\gamma c\beta+\frac{c\alpha s\beta s\varphi-c\beta s\beta c\gamma c\varphi}{s\beta}$$

$$\alpha_{13}^r=c\alpha c\gamma+$$
$$+\frac{c\alpha c^2\beta c\gamma c\varphi+c\beta c^2\gamma s\varphi-c\alpha c\gamma c\varphi+c^2\alpha c\beta s\varphi}{s^2\beta}$$

$$\alpha_{23}^r=c\beta c\gamma-\frac{s\beta c\beta c\gamma c\varphi}{s\beta}-\frac{s\beta c\alpha s\varphi}{s\beta}$$

$$\alpha_{33}^r=c^2\gamma-\frac{c\alpha c\beta c\gamma s\varphi}{s^2\beta}+\frac{c^2\beta c^2\gamma c\varphi}{s^2\beta}+$$
$$+\frac{c\beta c\gamma c\alpha s\varphi+c^2\alpha c\varphi}{s^2\beta}.$$

If the (Δ) axis versor is written down (u),

$$\left(u\right)=\left[c\alpha \quad c\beta \quad c\gamma\right]^T=$$
$$=\left[u_x \quad u_y \quad u_z\right]^T \text{ and } v\varphi=1-c\varphi, \tag{24}$$

we get the form given in eq.(2)

$$\left[a_r\right]^T=\begin{bmatrix} u_xu_xv\varphi+c\varphi & u_yu_xv\varphi-u_zs\varphi & u_zu_xv\varphi+u_ys\varphi \\ u_xu_yv\varphi+u_zs\varphi & u_yu_yv\varphi+c\varphi & u_zu_yv\varphi-u_xs\varphi \\ u_xu_zv\varphi-u_ys\varphi & u_yu_zv\varphi+u_xs\varphi & u_zu_zv\varphi+c\varphi \end{bmatrix}. \tag{25}$$

Having the rotation matrix, we can determ the direction cosines and the rotation angle:

$$\varphi = arccos\frac{\alpha_{11} + \alpha_{22} + \alpha_{33} - 1}{2}; \quad (26)$$

$$(u) = \frac{1}{2s\varphi}\begin{bmatrix} \alpha^r_{23} - \alpha^r_{32} \\ \alpha^r_{31} - \alpha^r_{13} \\ \alpha^r_{12} - \alpha^r_{21} \end{bmatrix}. \quad (27)$$

3. Conclusions

Using the method presented in the previous section the next conclusions result:

a) The presented solution is proper to the calculation of the φ angle with values between $0°$ and $180°$. If $\varphi = 0°$ or $180°$, the solution is undefined regarding the position of the rotation axis;

b) When the vector (u) isn't a unitary vector it must be normalized, finding in such a way its components face-to-face by the fixedly reference system.

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