THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
FASCICLE XIV MECHANICAL ENGINEERING, ISSN 1224-5615
2008

DYNAMIC ANALYSIS OF MECHANICAL SYSTEMS. CALCULUS OF THE EQUIVALENT DYNAMIC FORCES AND TORQUES

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ABSTRACT

This article presents the physical and mathematical models of mechanical systems with elastical and/or rigid elements, subjected to dynamical loads: forces and torques. The aim of the study is to establish some formulas easy to use to equate these dynamical loads in any point of the system. It had taken into account both the ideal mechanical systems (with no energy losses) and the real systems according to their mechanical efficiency.

KEYWORDS: mechanical systems, dynamics, equivalent forces and torques

1. Introduction

Trough dynamical loads it is understood the sum of the forces and couples of forces which operate on the system. These loads can be produced by:

-the driving motor(s); in this case, the load produces motor mechanical work

-the working devices of the equipments, the mechanical work being negative (required work)

-resistance loads like friction forces and torques; in this case the work is also negative (work of resistance/friction)

The method to reduce mechanical system and their loads to less complex systems consists in finding some mechanical features equivalent to primary ones. Being equivalent means for mechanical systems to have the same energy, power, to do the same work, to have the same natural frequencies and eigenvalues, aso. This article uses in order to equate forces and torques, the principle of the equivalency between the real forces and torques work and the work of the reduced/equivalent ones.

2. The general problem of forces and torques equation

It is considered a mechanical system subjected to n concentrated forces $\overline{F_i}$ $i=\overline{1,n}$ and m torques $\overline{M}_j$ $j=\overline{1,m}$. Also it is considered that infinitesimal displacements of the points of application of the forces are $d\overline{r_i}$ $i=\overline{1,n}$ and the infinitesimal rotations of the couples are $d\overline{\varphi}_j$, where $j=\overline{1,m}$.

The total infinitesimal mechanic work of all forces and torques is the sum of the infinitesimal mechanic work of each of them:

$$dL=\sum_{i=1}^{n}\overline{F_i}d\overline{s_i}+\sum_{j=1}^{m}\overline{M}_j d\overline{\varphi}_j \qquad (1)$$

Considering that the displacements of the points of application of the forces are parallel to the forces and the rotations are in a perpendicular plane to the couples, each scalar product of the relation (1) is equal with the product of the scalars, therefore the total infinitesimal mechanic work becomes:

$$dL=\sum_{i=1}^{n}F_i ds_i+\sum_{j=1}^{m}M_j d\varphi_j \qquad (2)$$

The products of each sum of relation (2) are positive for the motor work and negative for the mechanical work of resistance.

If it is necessary to calculate an equivalent force for the entire system, this force has to do the same mechanical work, meaning

$$\overline{F}_{eqv} d\overline{s} = dL, \tag{3}$$

where $d\overline{s}$ is vectorial displacement of the point of application of these forces.

Considering that the displacement of the point of application is parallel to the force, the relation (2) of the mechanical work becomes:

$$F_{eqv} ds = \sum_{i=1}^{n} F_i ds_i + \sum_{j=1}^{m} M_j d\varphi_j \tag{4}$$

If the relation (4) is divided by the infinitesimal time of motion dt, the work is now

$$F_{eqv} \frac{ds}{dt} = \sum_{i=1}^{n} F_i \frac{ds_i}{dt} + \sum_{j=1}^{m} M_j \frac{d\varphi_j}{dt}, \tag{5}$$

or

$$F_{eqv} v = \sum_{i=1}^{n} F_i v_i + \sum_{j=1}^{m} M_j \omega_j, \tag{6}$$

where v is the speed of the equivalent force (where it is making the equation of the entire system of loads)

v_i - the speed of the point of application of the force $\overline{F}_i$ $i = \overline{1,n}$

ω_j - the angular speed of the torque $\overline{M}_j$ $j = \overline{1,m}$

Dividing the relation (6) by v, the calculus formula for the equivalent force is:

$$F_{eqv} = \sum_{i=1}^{n} F_i \frac{v_i}{v} + \sum_{j=1}^{m} M_j \frac{\omega_j}{v} \tag{7}$$

To calculate an equivalent torque for the entire system of loads, this torque has to do the same infinitesimal mechanical work

$$\overline{M}_{eqv} d\overline{\varphi} = dL, \tag{8}$$

or

$$M_{eqv} d\varphi = dL \tag{9}$$

if the infinitesimal rotation vector is parallel to the torque vector. In this case the work becomes:

$$M_{eqv} d\varphi = \sum_{i=1}^{n} F_i ds_i + \sum_{j=1}^{m} M_j d\varphi_j \tag{10}$$

Dividing the relation (10) by dt, the infinitesimal mechanical work can be written as follows

$$M_{eqv} \cdot \omega = \sum_{i=1}^{n} F_i \frac{ds_i}{dt} + \sum_{j=1}^{m} M_j \frac{d\varphi_j}{dt}, \tag{11}$$

where $\frac{d\varphi}{dt} = \omega$ is the rotational speed of the equivalent torque M_{eqv}.

By dividing the work (11) through ω, it obtains the calculus formula for equivalent torque like this:

$$M_{eqv} = \sum_{i=1}^{n} F_i \frac{v_i}{\omega} + \sum_{j=1}^{m} M_j \frac{\omega_j}{\omega} \tag{12}$$

3. The equivalent torque of the mechanical transmissions with gears

It considers a simple mechanical transmission with one step gear like in fig. 1.

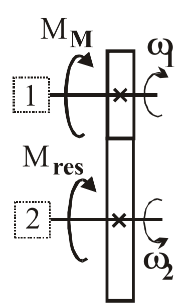

Fig. 1 The calculus model of the one step gearing mechanical transmission
1-driving shaft
2-driven shaft
$\mathbf{M_M}$-the moment/torque of the motor
$\mathbf{M_{res}}$-the resistance moment/torque
$\boldsymbol{\omega_1}$, $\boldsymbol{\omega_2}$-rotational speed of the shafts

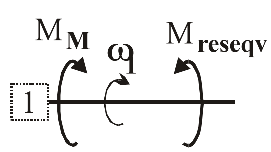

Fig. 2 The calculus diagram for the equivalent moment of resistance

For the gear transmission from fig. 1,

the ratio is:

$$i = i_{12} = \frac{\omega_1}{\omega_2} \tag{13}$$

Figure 2 shows the calculus diagram of the equivalent moment of resistance only (the calculus diagram and the way to calculate the equivalent inertia are acc. to [1]). The calculus diagram for the equivalent motor moment is shown in the fig. 3.

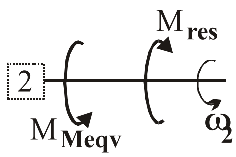

Fig. 3 The calculus diagram for the equivalent driving moment

3.1. Ideal gear transmission

If there are no losses in the step gear, the equivalent torques can be calculated by the relation (12) as follows:

-the equivalent moment of resistance (calculated to the shaft no. **1**)

$$M_{reseqv} = M_{res}\frac{\omega_2}{\omega_1} = \frac{M_{res}}{i} \tag{14}$$

-the equivalent motor moment (calculated to the shaft no. **2**)

$$M_{Meqv} = M_M\frac{\omega_1}{\omega_2} = M_M \cdot i \tag{15}$$

Since the ratio i can take different values, we may draw some intermediate conclusions:

1^0if $i = 1$ (gears for changing the sense of rotation only) the both equivalent moment and the equivalent moment of resistance stays unchanged

2^0if $i > 1$ (reduced step gear) the motor moment is amplified by i and the moment of resistance is divided/decreased by i

3^0if $i < 1$ (amplifier step gear) the motor moment is divided by i and the moment of resistance is amplified by i.

3.2. Gear with mechanical losses

To establish the analytical formulas of the equivalent moments for a transmission with one step gear with mechanical losses. It considers the diagram from fig. 1, where the ratio is done by (13) and the mechanical efficiency is η. The motor moment is in the sense of $\overline{\omega}_1$ and the moment of resistance is against the sense of $\overline{\omega}_2$.

For the diagram shown in fig. 2, the power balance can be written

$$-M_{reseqv}\omega_1 = -M_{res}\omega_2 - P_{loss}, \tag{16}$$

where P_{loss} is the loss of power in the gear.

If the loss of power is written as

$$P_{loss} = M_{loss}\omega_1, \tag{17}$$

where M_{loss} is the loss of moment in the gear, relation (16) becomes:

$$(M_{reseqv} - M_{loss})\omega_1 = M_{res}\omega_2, \tag{18}$$

or

$$M_{reseqv} - M_{loss} = M_{res}\frac{\omega_2}{\omega_1} \tag{19}$$

In order to underline the influence of the mechanical efficiency of the gear, it divides (19) by M_{reseqv}, obtaining:

$$\frac{M_{reseqv} - M_{loss}}{M_{reseqv}} = \frac{M_{res}}{M_{reseqv}}\frac{\omega_2}{\omega_1} \tag{20}$$

Observing that in the left side of the equal sign is the mechanical efficiency η, the relation (20) becomes

$$\eta = \frac{M_{res}}{M_{reseqv}}\frac{\omega_2}{\omega_1}, \tag{21}$$

from which it is obtained the expression of the equivalent moment of resistance on the driving shaft **1**:

$$M_{reseqv} = \frac{M_{res}}{\eta}\frac{\omega_2}{\omega_1} = \frac{M_{res}}{i\eta} \tag{22}$$

For equivalent motor moment calculation it considers the diagram from fig. 3. In this case, the power balance is as follows:

$$M_{Meqv}\omega_2 = M_M\omega_1 - P_{loss} \tag{23}$$

Taking in consideration the relation (17), the power balance can be written

$$M_{Meqv}\omega_2 = (M_M - M_{loss})\omega_1, \tag{24}$$

or like this:

$$M_{Meqv}\omega_2 = \frac{M_M - M_{loss}}{M_M} M_M \omega_1 \quad (25)$$

Since the fraction from the right side of the equation (25) is the mechanical efficiency η, the power balance can be written

$$M_{Meqv}\omega_2 = \eta M_M \omega_1, \quad (26)$$

from which it is obtained the equivalent motor moment on the driven shaft **2**:

$$M_{Meqv} = \eta M_M \frac{\omega_1}{\omega_2} = M_M i\eta \quad (27)$$

As the mechanical efficiency $\eta < 1$, it may say that, taking into consideration the mechanical losses from the gear leads to:
1^0a decreasing of the equivalent motor moment
2^0an increasing of the equivalent moment of resistance

4. The equivalent torques calculus for the driving mechanisms with gears

To understand the equation method for the torques, it is considered a simple electric mechanical driving of a working device like in fig. 4. It is assumed that the acting torques are:
-the driving/motor moment M_M
-the moment of resistance M_{ir}, $i = \overline{1,4}$
-the moment of the working device M_{WD}

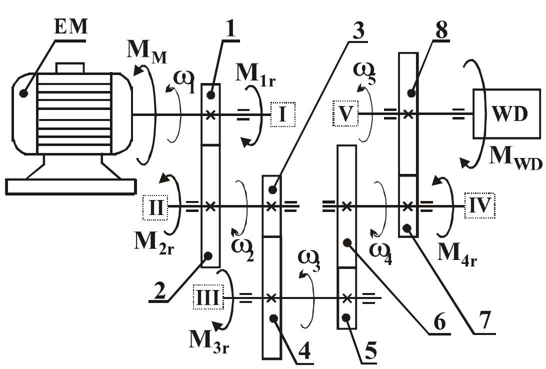

Fig. 4 The skeleton diagram
of a driving mechanism with four steps gear
EM-electromotor
WD-working device
1,2,3,4,5,6,7,8-gear wheels (*gw*)
I-driving shaft
V-driven shaft
II,III,IV-intermediate motion shaft

Considering that the transmission shafts are rigid, the ratio of each step gear is:
-gear ***gw1-gw2***

$$i_1 = i_{12} = \frac{\omega_{gw1}}{\omega_{gw2}} = \frac{\omega_1}{\omega_2} \quad (28)$$

-gear ***gw3-gw4***

$$i_2 = i_{34} = \frac{\omega_{gw3}}{\omega_{gw4}} = \frac{\omega_2}{\omega_3} \quad (29)$$

-gear ***gw5-gw6***

$$i_3 = i_{56} = \frac{\omega_{gw5}}{\omega_{gw6}} = \frac{\omega_3}{\omega_4} \quad (30)$$

-gear ***gw7-gw8***

$$i_4 = i_{78} = \frac{\omega_{gw7}}{\omega_{gw8}} = \frac{\omega_4}{\omega_5} \quad (31)$$

4.1. The equivalent torques calculus on the motor shaft

If it’s necessary an operation of equation of all torques on the driven shaft **I**, the calculus diagram is shown in fig. 5. In this case, it has to equate all the torques excepting

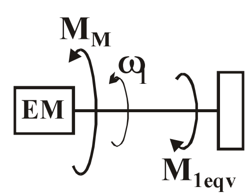

Fig. 5 The calculus diagrams for the equivalent moment on the driving shaft **I**

the motor moment M_M.

Neglecting the mechanical losses in the gear, the power balance is as follows:

$$-M_{1eqv}\omega_1 = -M_{1res}\omega_1 - M_{2res}\omega_2 - \\ - M_{3res}\omega_3 - M_{4res}\omega_4 - M_{WD}\omega_5 \quad (32)$$

where all minus signs are done by the differences between the senses of the torques and the senses of the rotation speed.

If the relation (32) is multiplied by “-1” and divided by ω_1, it is obtained the sum of the equivalent torques on shaft **1** like this:

$$M_{1eqv} = M_{1res} + M_{2res}\frac{\omega_2}{\omega_1} + M_{3res}\frac{\omega_3}{\omega_1} + M_{4res}\frac{\omega_4}{\omega_1} + M_{WD}\frac{\omega_5}{\omega_1} \quad (33)$$

Taking into consideration the definition of the gear ration done by the relations (28)-(31), the total equivalent moment on the motor shaft **I** can be written as follows:

$$M_{1eqv} = M_{1res} + \frac{M_{2res}}{i_1} + \frac{M_{3res}}{i_1 i_2} + \frac{M_{4res}}{i_1 i_2 i_3} + \frac{M_{WD}}{i_1 i_2 i_3 i_4} \quad (34)$$

In order to determine the calculus formula of the equivalent torques on the motor shaft taking into consideration the mechanical efficiency of the gears (η_1, η_2, η_3, η_4), it uses the relation (22) because all the equivalent torques are resistant. In this case, the total equivalent torque on the motor shaft is:

$$M_{1eqv} = M_{1res} + \frac{M_{2res}}{i_1\eta_1} + \frac{M_{3res}}{i_1 i_2 \eta_1 \eta_2} + \frac{M_{4res}}{i_1 i_2 i_3 \eta_1 \eta_2 \eta_3} + \frac{M_{WD}}{i_1 i_2 i_3 i_4 \eta_1 \eta_2 \eta_3 \eta_4} \quad (35)$$

4.2. The equivalent torques calculus on the working device shaft

Depending of the type of dynamical analysis of the mechanical transmission with gear, it can equate all the moments on the working device shaft. In this case, the simplified diagram is shown in fig. 6.

Neglecting the mechanical losses from gears, the power balance can be written like this:

$$M_{5eqv}\omega_5 = -M_{4res}\omega_4 - M_{3res}\omega_3 - M_{2res}\omega_2 - M_{1res}\omega_1 + M_M\omega_1 \quad (36)$$

If the relation (36) is divided by ω_5, it is obtained the calculus formula for the total equivalent moment on the working device shaft **V** as follows:

$$M_{5eqv} = -M_{4res}\frac{\omega_4}{\omega_5} - M_{3res}\frac{\omega_3}{\omega_5} - M_{2res}\frac{\omega_2}{\omega_5} - M_{1res}\frac{\omega_1}{\omega_5} + M_M\frac{\omega_1}{\omega_5} \quad (37)$$

Function of four ratios of the gears done by the relations (28)-(31), the relation (37) can be written like this:

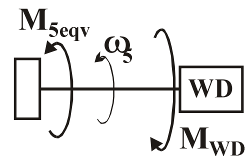

Fig. 6 The calculus diagram for the equivalent moment on the driven shaft **V**

$$M_{5eqv} = (M_M - M_{1res})i_1 i_2 i_3 i_4 - M_{2res} i_2 i_3 i_4 - M_{3res} i_3 i_4 - M_{4res} i_4 \quad (38)$$

Now, if we consider the losses of power in the gears by the mechanical efficiency of the gears, the calculus formula of the total equivalent moment on the working device gear **V** becomes:

$$M_{5eqv} = M_M i_1 i_2 i_3 i_4 \eta_1 \eta_2 \eta_3 \eta_4 - M_{1res}\frac{i_1 i_2 i_3 i_4}{\eta_1 \eta_2 \eta_3 \eta_4} - M_{2res}\frac{i_2 i_3 i_4}{\eta_2 \eta_3 \eta_4} - M_{3res}\frac{i_3 i_4}{\eta_3 \eta_4} - M_{4res}\frac{i_4}{\eta_4} \quad (39)$$

4.3. The equivalent torques calculus on intermediate motion shafts

If the equation of the moments has to be done on another shaft (than motor shaft or working device shaft), the process of equation will be done taking into consideration the angular speed of that shaft and the branch of the skeleton of the transmission.

For instance, if the torque equation is

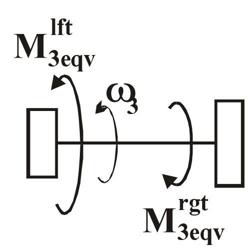

Fig. 7 The calculus diagram for the equivalent moment on the intermediate motion shaft **III**

done on the intermediate shaft **III**, the calculus diagram is that from fig. 7. The equivalent torques from that diagram are composed from:

M_{3eqv}^{lft} - the sum of the equivalent moments of the left branch of the shaft **III** (motor moment, resistant moments from the shafts **I**, **II** and **III**)

M_{3eqv}^{rgt} - the sum of the equivalent moments of the right branch of the shaft **III** (working device moment, resistant moment from the shaft **IV**)

Considering the nature of the moments (motor, resistant), the power balance of the left branch on the shaft **III** is:

$$M_{3eqv}^{lft}\omega_3 = -M_{3res}\omega_3 - M_{2res}\omega_2 - M_{1res}\omega_1 + M_M\omega_1 \quad (40)$$

The calculus formula of the total equivalent moment on left branch is obtained by dividing the relation (40) through ω_3:

$$M_{3eqv}^{lft} = -M_{3res} - M_{2res}\frac{\omega_2}{\omega_3} - M_{1res}\frac{\omega_1}{\omega_3} + M_M\frac{\omega_1}{\omega_3} \quad (40)$$

The relation giving M_{3eqv}^{lft} can be written taking into consideration the gears ratio as follows:

$$M_{3eqv}^{lft} = (M_M - M_{1res})i_1 i_2 - M_{2res}i_2 - M_{3res} \quad (40)$$

The power balance for the right branch of the transmission is

$$-M_{3eqv}^{rgt}\omega_3 = -M_{4res}\omega_4 - M_{WD}\omega_5, \quad (41)$$

or, by dividing trough "$-\omega_3$":

$$M_{3eqv}^{rgt} = M_{4res}\frac{\omega_4}{\omega_3} + M_{WD}\frac{\omega_5}{\omega_3} \quad (42)$$

Relation (42) can be written function of gear ratio as follows:

$$M_{3eqv}^{rgt} = \frac{M_{4res}}{i_3} + \frac{M_{5res}}{i_3 i_4} \quad (43)$$

For the equation of the moments taking into consideration the losses of power by the mechanical efficiency of the gear (η_1, η_2, η_3, η_4), it has to use relations (22) and (27). In this case, the equivalent total moments on the shaft **III** are:

-on left branch

$$M_{3eqv}^{lft} = M_M i_1 i_2 \eta_1\eta_2 - M_{1res}\frac{i_1 i_2}{\eta_1\eta_2} - M_{2res}\frac{i_2}{\eta_2} - M_{3res} \quad (44)$$

-on right branch

$$M_{3eqv}^{rgt} = \frac{M_{4res}}{i_3\eta_3} + \frac{M_{WD}}{i_3 i_4 \eta_3\eta_4} \quad (45)$$

5. Conclusions

This article describes the general method of equation of statical or dynamic loads applied to mechanical systems. Particularly, there are the proceeding and the formulae for the equivalent torques' calculus of the mechanical transmission with gears.

These methods and calculus formulae are useful both to the design engineers and to the dynamics experts. Also, the methods to equate the torques can be useful to the students, candidates for master's and doctor's degree.

6. References

[1]**Debeleac, C.N., Drăgan, N.** – *"The dynamic modelling of the mechanical systems. Calculus of the equivalent mass and equivalent mass inertia"*, The Annals of "Dunărea de Jos" University of Galati, Fascicle XIV Mechanical Engineering, Galati, 2007