THE ANNALS OF “DUNAREA DE JOS” UNIVERSITY OF GALATI
FASCICLE XIV MECHANICAL ENGINEERING, ISSN 1224-5615
2008

ABOUT THE FREE PARAMETER VIBRATIONS OF THE MECHANICAL SYSTEMS

Assoc.Prof.Dr.Eng. Gh. Cautes
"Dunarea de Jos" University of Galati

ABSTRACT

The oscillation movement of a mechanical non-linear system is not easy to solve exactly in an analytical way. The approximate solutions are based on different methods and give different values with different approximation degree. In this paper it is shown that for such differential equations, such as $m\ddot{x} - c \cdot t^k \cdot x = 0$ *which describe free parametric vibrations of some elastic systems, there can be found analytical or approximate solutions. Using the substitution* $\dot{x} = x \cdot y$*, y=y(t), the differential equation becomes a Riccati special equation for which, using the Bessel functions, we obtain analytical or approximate solutions.*

KEYWORDS: Mechanical, System, Oscillation, Equation, Non-linear.

1. Introduction

Free oscillations of the non-linear mechanical systems are described by differential equations such as

$$m\ddot{x} + F_i = 0 \tag{1}$$

where x is the movement, m is the mass of the system and F_i are the internal forces (elastic and damping).

The free parametric vibrations, without dumping, of some elastic systems are described by differential equations, such as [1]

$$m\ddot{x} - c \cdot t^k \cdot x = 0 \tag{2}$$

where t is the time, c and k are real invariables.

After division to m, the mass doesn’t appear any longer in the equation.

For the first differential equation, it doesn’t exist any method for calculating the precise analytical solutions but for the particular ones.

Using the substitution

$$\dot{x} = x \cdot y,\ \text{y=y(t)}, \tag{2'}$$

the last equation becomes the special Riccati differential equation.

We now show that we can find ,for this kind of equation (and also the differential equation(2)) analytical exact or aproximate solutions, using the Bessel functions.

2. Teoretical concept

In technology, physics, astronomy we can find applications of the differential equation of the second order

$$x^2 \cdot y'' + x \cdot y' + \left(x^2 - p^2\right) \cdot y = 0 \tag{3}$$

known as the Bessel equation.

If p is not a whole number and it is not half of a whole number, than the general integer of the equation (3) is [2]

$$y = c_1 \cdot J_p(x) + c_2 \cdot J_{-p}(x)\ ; \quad c_1, c_2 \in R \tag{4}$$

where

$$J_p(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(p+s+1)} \cdot \left(\frac{x}{2}\right)^{p+2s} \tag{5}$$

is the Bessel function and Γ is the Euler function.

If p=n is a whole pozitive number, then

$$J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\cdot\left(\frac{x}{2}\right)^{n+2s} \qquad (6)$$

is a solution for the equation (3) with

$$K_n(x)=\beta\cdot J_n(x)\cdot\ln x+x^{-n}\sum_{s=0}^{\infty}\beta_s x^s \qquad (7)$$

where $\beta,\beta_0,\beta_1,\ldots\in R.$

The general integer of the equation (3) will be

$$y=c_1\cdot J_n(x)+c_2K_n(x)\ ;\quad c_1, c_2\in R. \qquad (8)$$

If p is half of a whole odd number, $p=\frac{2n+1}{2}$, the general integer of the equation (3) is like (4), where

$$J_{\frac{2n+1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[P_n\left(\frac{1}{x}\right)\sin x+Q_n\left(\frac{1}{x}\right)\cos x\right] \qquad (9)$$

with $P_n\left(\frac{1}{x}\right)$, $Q_n\left(\frac{1}{x}\right)$ polynomial in $\frac{1}{x}$.

Adding a new independent variable t and a new function u [3]

$$y=t^{\alpha}\cdot u\ ;\ x=\gamma\cdot t^{\beta} \qquad (10)$$

with $\alpha,\beta,\gamma\in R$ which are not null, we obtain

$$\frac{dt}{dx}=\frac{1}{\beta\gamma}t^{1-\beta}\ ;$$

$$\frac{dy}{dx}=\frac{1}{\beta\gamma}t^{1-\beta}\cdot\frac{dy}{dt}\ ;$$

$$\frac{d^2y}{dx^2}=\frac{1}{\beta\gamma}t^{1-\beta}\left(\frac{1}{\beta\gamma}t^{1-\beta}\frac{d^2y}{dt^2}+\frac{1-\beta}{\beta\gamma}\cdot t^{-\beta}\cdot\frac{dy}{dt}\right);$$

$$\frac{dy}{dt}=t^{\alpha}\cdot\frac{du}{dt}+\alpha\cdot t^{\alpha-1}\cdot u\ ; \qquad (11)$$

$$\frac{d^2y}{dt^2}=t^{\alpha}\cdot\frac{d^2u}{dt^2}+2\alpha\cdot t^{\alpha-1}\cdot\frac{du}{dt}+\alpha(\alpha-1)\cdot t^{\alpha-2}\cdot u$$

Replacing y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in the differential equation (3) and $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ in u, $\frac{du}{dt}$, $\frac{d^2u}{dt^2}$ than

$$t^2\cdot\frac{d^2u}{dt^2}+(2\alpha+1)\cdot t\cdot\frac{du}{dt}+ +\left(\alpha^2-\beta^2p^2+\beta^2\gamma^2\cdot t^{2\beta}\right)\cdot u=0. \qquad (12)$$

Because the differential equation (3) has the general integer (4), the equation (12) will have

$$u=t^{-\alpha}\left[c_1J_p\left(\gamma\cdot t^{\beta}\right)+c_2J_{-p}\left(\gamma\cdot t^{\beta}\right)\right] \qquad (13)$$

In which we replace $J_{-p}\left(\gamma\cdot t^{\beta}\right)$ with $K_n\left(\gamma\cdot t^{\beta}\right)$ if p=n is zero or a whole and pozitive number.

The differential equation (12) is

$$t^2\cdot\frac{d^2u}{dt^2}+a\cdot t\cdot\frac{du}{dt}+\left(b+c\cdot t^m\right)\cdot u=0, \qquad (14)$$

for

$$(2\alpha+1)=a,\ \alpha^2-\beta^2p^2=b,$$

$$\beta^2\gamma^2=c,\ 2\beta=m. \qquad (15)$$

For any equation like (14) where the invariable c and m are not null we can find using (15) the values of α,β,γ,p and the general integer of the differential equation (12) will be expressed using the Bessel functions after formulas (4) and (8).

We shall now take the special Riccati equation [4]

$$y'+k_1\cdot y^2-k_2\cdot x^k=0 \qquad (16)$$

with $k,k_1,k_2\in R$, about which we know that we can only find solutions in some special situations.

Using the substitution

$$k_1\cdot y=\frac{1}{u}\cdot\frac{du}{dx} \qquad (17)$$

The differential equation (16) becomes

$$x^2 \cdot \frac{d^2u}{dx^2} - k_1 k_2 x^{k+2} \cdot u = 0 \tag{18}$$

because with the derivation of the substitution (17) we obtain

$$k_1 \cdot y' = -\frac{1}{u^2} \cdot \left(\frac{du}{dx}\right)^2 + \frac{1}{u} \cdot \frac{d^2u}{dx^2} \cdot \tag{19}$$

The equation (18) is like (13) where

$$a=b=0,\ c=-k_1 \cdot k_2,\ m=k+2 \tag{20}$$

and its solutions are Bessel functions if the invariable k_1,k_2 have opposite signs for $c \geq 0$, as (15).

After solving the function u from the equation (18) with (17) we can find the general soluton y of the equation (16).

3. Examples

1. The vibrations of a mechanical system are described by the differential equation

$$\ddot{x} + \frac{2}{9} \cdot x \cdot t^{-4} = 0 \tag{21}$$

which is equal with the special Riccati differential equation

$$y' = \frac{1}{3} \cdot y^2 + \frac{2}{3} \cdot x^{-4} \tag{22}$$

such as (16) where

$$k_1 = -\frac{1}{3},\ k_2 = \frac{2}{3},\ k = -4 \tag{23}$$

and that becomes (21) using the substitution (17), we cand find the movement function x(t).

The differential equation (21), with

$$a = b = 0; c = \frac{2}{9}, \tag{24}$$

and from (15) we obtain

$$\alpha = -\frac{1}{2},\ p = \frac{1}{2},\ \beta = -1,\ \gamma = \frac{\sqrt{2}}{3},\ m = -2 \cdot \tag{25}$$

The invariable p is half of a whole odd number,than $n = 0$ and equation (21) has the general solution

$$x(t) = t^{\frac{1}{2}}\left[c_1 \cdot J_{\frac{1}{2}}\left(\frac{\sqrt{2}}{3t}\right) + c_2 \cdot J_{-\frac{1}{2}}\left(\frac{\sqrt{2}}{3t}\right)\right] = \tag{26}$$
$$= t\sqrt{\frac{6}{\pi\sqrt{2}}}\left(c_1 \cdot \sin\frac{\sqrt{2}}{3t} + c_2 \cdot \cos\frac{\sqrt{2}}{3t}\right),$$

with

$$J_{\frac{1}{2}}(\mathrm{v}) = \sqrt{\frac{2}{\pi \mathrm{v}}} \sin \mathrm{v}\ , \tag{27}$$
$$J_{-\frac{1}{2}}(\mathrm{v}) = \sqrt{\frac{2}{\pi \mathrm{v}}} \cos \mathrm{v}.$$

Derivating $x(t)$ we obtain

$$x'(t) = \sqrt{\frac{6}{\pi\sqrt{2}}}\left(c_1 \sin\frac{\sqrt{2}}{3t} + c_2 \cos\frac{\sqrt{2}}{3t}\right) + \tag{28}$$
$$+ \frac{\sqrt{2}}{3t}\left(c_1 \sin\frac{\sqrt{2}}{3t} - c_2 \cos\frac{\sqrt{2}}{3t}\right)$$

and using the substitution (17) we obtain the general solution of the Riccati differential equation (22)

$$y(x) = -3 \cdot x^{-1} + \sqrt{2} \cdot x^{-2} \cdot \frac{c_0 \cdot \cos\frac{\sqrt{2}}{3x} - \sin\frac{\sqrt{2}}{3x}}{c_0 \cdot \sin\frac{\sqrt{2}}{3x} + \cos\frac{\sqrt{2}}{3x}} =$$
$$= -\frac{3}{x} + \frac{\sqrt{2}}{x^2} \cdot \frac{c_0 - tg\,\frac{\sqrt{2}}{3x}}{c_0 tg\,\frac{\sqrt{2}}{3x} + 1}, \tag{29}$$

where

$$c_0 = \frac{c_1}{c_2} = const.$$

2. The vibrations of a mechanical system are described by the differential equation

$$\ddot{x}-\frac{1}{4}\cdot x\cdot t^{-5}=0 \qquad (30)$$

which is equal with the special Riccati equation

$$y' = \frac{1}{2}\cdot y^2+\frac{1}{2}\cdot x^{-5} \qquad (31)$$

such as (16) where

$$k_1=-\frac{1}{2},\, k_2=\frac{1}{2},\, k=-5 \qquad (32)$$

and that becomes (30) using the substitution (17), we cand find the movement function x(t).

The differential equation (30), with

$$a=b=0; c=\frac{1}{4}, \qquad (33)$$

and from (15) we obtain

$$\alpha=-\frac{1}{2},\, p=\frac{3}{2},\, \beta=-\frac{2}{3},\, \gamma=\frac{3}{4},\, m=\frac{4}{3}. \qquad (34)$$

The invariable p is half of a whole odd number,than $n=1$ and equation (30) has the general solution

$$x(t)=t^{\frac{1}{2}}\left[c_1\cdot J_{\frac{3}{2}}\left(\frac{3}{4}t^{\frac{2}{3}}\right)+c_2\cdot J_{-\frac{3}{2}}\left(\frac{3}{4}t^{\frac{2}{3}}\right)\right]=$$

$$=2\sqrt{\frac{2}{3\pi}}t^{\frac{1}{6}}\left[c_1\left(\frac{4}{3}t^{-\frac{2}{3}}\sin\frac{3}{4}t^{\frac{2}{3}}-\cos\frac{3}{4}t^{\frac{2}{3}}\right)\right]-$$

$$-2\sqrt{\frac{2}{3\pi}}t^{\frac{1}{6}}\left[c_2\left(\sin\frac{3}{4}t^{\frac{2}{3}}+\frac{4}{3}t^{-\frac{2}{3}}\cos\frac{3}{4}t^{\frac{2}{3}}\right)\right], \qquad (35)$$

with

$$J_{\frac{3}{2}}(\mathrm{v})=\sqrt{\frac{2}{\pi \mathrm{v}}}\left(\frac{1}{v}\sin v - \cos v\right),$$

$$(36)$$

$$J_{-\frac{3}{2}}(\mathrm{v})=-\sqrt{\frac{2}{\pi \mathrm{v}}}\left(\sin v+\frac{1}{v}\cos \mathrm{v}\right)$$

and $c_1, c_2 = const$.

Derivating $x(t)$ and using the substitution (17) we obtain the general solution of the Riccati differential equation.

4. Conclusions.

1. For nay non-linear differential equation such as (1) we can obtain an analytical exact or aproximate solution using the Bessel functions.
2. For any special Riccati differential equation we can find an analytical exact or aproximate solution.

References

[1] **Cautes, Gh.** *Ecuatii diferentiale. Notiuni teoretice. Aplicatii in mecanica si electricitate.* Ed. Academica, ISBN 978-973-8937-08-06, Galati, 2006.
[2] **Olariu, V., Prepelita, V.** *Matematici speciale*, EDP, Bucuresti, 1985
[3] **O'Neil** *Advanced Engineering Mathematics*, King's College London, 1984
[4] **Cautes, Gh.** *Matematici speciale,* Ed. Tehnica Info, ISBN 978-9975-910-69-9, Chisinau, 2008