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THE CALCULUS OF THE EQUIVALENT RIGIDITY COEFFICIENTS FOR THE SHAFTS OF THE ELASTICAL SYSTEMS

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ABSTRACT

The components of the mechanical systems like machines and/or technological equipment have a certain elasticity or rigidity. Knowing the rigidity of each component is very important for the studies whose goal is to establish the dynamic efforts and the stresses in each shaft, gear wheel, steel structure, aso. The rigidity can be an important factor for the dynamic load estimation process. The components with high elasticity are the most important inducers of elastical forces and couples of force; we can enumerate: shafts, coupling gears, gears, elastical couplings, springs, long steel structures, some working devices, aso. This article presents a method and an equation involving the rigidities of the elastical shafts of the mechanical transmissions with gears in any point of the system, so that the dynamic analysis should become easier.

KEYWORDS: elastical systems, equivalent rigidity, gearing, shaft stress

1. Introduction

The equivalent coefficient of rigidity is the mechanical feature of an equivalent elastical element (generally named spring), which replaces the real element on the basic principle of the equation of the potential energy [1]. This means that the deformation potential energy of the equivalent element V_{eqv} is equal to the deformation potential energy of the actual element V.

2. Calculus of the equivalent rigidity of the shafts with one step gearing

In order to describe the rigidity equation method, it is considered a simple mechanical driving system as in fig.1, where M_M is the driving motor moment, M_{WD} is the moment of working device, **2** and **3** are the wheels of the one step gearing, k_1 and k_2 are the rigidity coefficients of the shaft **I** (driving shaft) respectively shaft **II** (driven shaft).

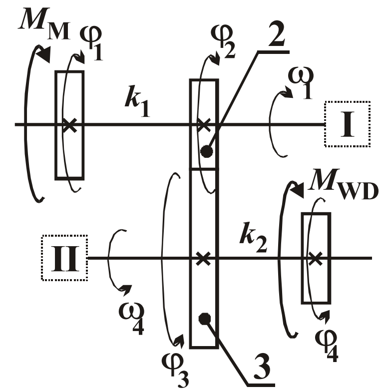

Fig. 1 The model to calculate the equivalent rigidities of the shafts

It is considered that the mechanical efficiency of the gearing is η and the ratio is

$$i = \frac{\omega_{gw2}}{\omega_{gw3}}, \qquad (1)$$

where ω_{gw2} and ω_{gw3} are the angular speeds of the wheel gear **2** respectively **3**.

The instantaneous real angular rotations of the shafts' terminations are:

-for the shaft **I** - φ_1, φ_2

-for the shaft **II** - φ_3, φ_4

The equivalent inertia moments of the working device and of wheel gear **3** can be calculated according to [2].

2.1. Calculus of the equivalent rigidity on the driving shaft

If the needs of equation is to be done on the shaft **I**, the fig. 2 shows the calculus model, where the significance of the notations is as follows:

► k_{2eqv} - the equivalent rigidity coefficient of the driven shaft

► φ_3^*, φ_4^* - the equivalent angular rotations of the driven shaft's terminations

► ω_1 - the average angular speed of the driving shaft (**in steady-state conditions**)

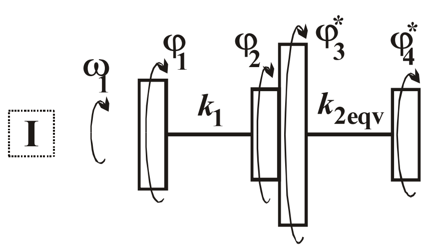

Fig. 2 The model to calculate the equivalent rigidity of the driven shaft

The deformation potential energy of the shaft **II** for real system (fig. 1) can be written as:

$$V = \frac{1}{2} k_2 (\varphi_3 - \varphi_4)^2 \qquad (2)$$

For the same shaft **II**, the potential energy, on the basis of the equivalent model from fig. 2, has the expression:

$$V_{eqv} = \frac{1}{2} k_{2eqv} \left(\varphi_3^* - \varphi_4^*\right)^2 \qquad (3)$$

Equating the expressions (2) and (3) of the potential energy of the shaft **II**, we obtain:

$$\frac{k_2}{k_{2eqv}} = \frac{\left(\varphi_3^* - \varphi_4^*\right)^2}{(\varphi_3 - \varphi_4)^2} \qquad (4)$$

Taking into consideration that, **in steady-state conditions**, the working device moment (of resistance) is equal to the elastical moment from the driven shaft, it may be written as follows:

● for real system

$$M_{WD} = k_2 (\varphi_3 - \varphi_4) \qquad (5)$$

● for equivalent system

$$M_{WDeqv} = k_{2eqv} \left(\varphi_3^* - \varphi_4^*\right) \qquad (6)$$

Dividing the relations (5) and (6), it is obtained:

$$\frac{\varphi_3^* - \varphi_4^*}{\varphi_3 - \varphi_4} = \frac{k_2}{k_{2eqv}} \frac{M_{WDeqv}}{M_{WD}} \qquad (7)$$

Considering the relation (4), it may be written

$$\sqrt{\frac{k_2}{k_{2eqv}}} = \frac{k_2}{k_{2eqv}} \frac{M_{WDeqv}}{M_{WD}} \qquad (8)$$

or

$$\frac{k_{2eqv}}{k_2} = \left(\frac{M_{WDeqv}}{M_{WD}}\right)^2 \qquad (9)$$

From (9), we can write:

$$k_{2eqv} = k_2 \left(\frac{M_{WDeqv}}{M_{WD}}\right)^2 \qquad (10)$$

In order to estimate the fraction between working device moments from (10), it has to be written the working device power both for real system and for equivalent system.

a)Ideal step gearing ($\eta = 1$)

If there are no mechanical losses in the gearing **2-3**, all power from the motor goes to the working device, that's why it may be written

$$P = M_{WDeqv} \omega_1 = M_{WD} \omega_4 \qquad (11)$$

From the relation (11), we can write the fraction between the working device equivalent moment and the working device real moment as follows:

$$\frac{M_{WDeqv}}{M_{WD}} = \frac{\omega_4}{\omega_1} \qquad (12)$$

Since, **<u>in steady-state conditions</u>**, the average angular speed of the working device ω_4 is equal to angular speed of the wheel gear **3** (ω_3) and the average angular speed of the motor ω_1 is equal to the angular speed of the wheel gear **2** (ω_2), the relation (12) may be written:

$$\frac{M_{WDeqv}}{M_{WD}} = \frac{\omega_4}{\omega_1} = \frac{\omega_3}{\omega_2} = \frac{1}{i} \qquad (13)$$

Taking into consideration relation (13), the calculus formula for the rigidity coefficient of the shaft **II** on the motor shaft is:

$$k_{2eqv} = \frac{k_2}{i^2} \qquad (14)$$

b)Step gearing with mechanical losses ($\eta < 1$)

If there are mechanical losses in the gearing **2-3**, the power from the motor goes partially to the working device and the difference is dissipated in the gearing. In this case, we may write

$$P = M_{WDeqv}\omega_1 = M_{WD}\omega_4 + P_{loss}, \qquad (15)$$

where P_{loss} is the power losses in the gearing.

If the loss of power is written function of ω_1 as

$$P_{loss} = M_{loss}\omega_1, \qquad (16)$$

where M_{loss} is the equivalent loss of moment, in the steady-state conditions ($\omega_1 = \omega_2$, $\omega_3 = \omega_4$), the power balance done by (15) can be written like

$$\left(M_{WDeqv} - M_{loss}\right)\omega_2 = M_{WD}\omega_3, \qquad (17)$$

or

$$\frac{M_{WDeqv} - M_{loss}}{M_{WDeqv}} \frac{M_{WDeqv}}{M_{WD}} = \frac{\omega_3}{\omega_2} \qquad (18)$$

Since, the left side of the relation (18) is the mechanical efficiency η of the gearing **2-3** and the fraction of the right side is the inverse of the gearing ratio (1), we may write

$$\frac{M_{WDeqv}}{M_{WD}} = \frac{1}{i\eta} \qquad (19)$$

Consequently, the relation (10) becomes:

$$k_{2eqv} = \frac{k_2}{i^2\eta^2} \qquad (20)$$

2.2. Calculus of the equivalent rigidity on the driven shaft

Figure 3 shows the calculus model of the rigidity equation on the axle of the driven shaft **II**. The significance of the notations is as follows:

► k_{1eqv} - the equivalent rigidity coefficient of the driving shaft

► φ_1^*, φ_2^* - the equivalent angular rotations of the driving shaft's terminations

► ω_4 - the average angular speed of the driving shaft (**in steady-state conditions**)

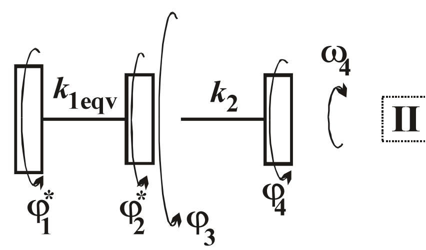

Fig. 3 The model to calculate the equivalent rigidity of the driving shaft

As in §2.1, the deformation potential energy of the driving shaft **I** can be written like this

$$V = \frac{1}{2}k_1\left(\varphi_1 - \varphi_2\right)^2 \qquad (21)$$

For the same shaft **I**, the potential energy calculated with the equivalent rigidity k_{1eqv} and angular deflections φ_1^*, φ_2^* is as follows

$$V_{eqv} = \frac{1}{2}k_{1eqv}\left(\varphi_1^* - \varphi_2^*\right)^2 \qquad (22)$$

Since the potential energy of the shaft **I** has to remain the same after the process of equation, from relations (21) and (22) it can be written the fraction between the rigidities as follows:

$$\frac{k_1}{k_{1eqv}} = \frac{\left(\varphi_1^* - \varphi_2^*\right)^2}{\left(\varphi_1 - \varphi_2\right)^2} \quad (23)$$

Assuming that, **<u>in steady-state conditions</u>**, the motor has the same average angular speed like the wheel gear **2** and the working device has the same average angular speed like the wheel gear **3**, that meaning $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$, the motor moment has to be equal to the elastical torsion moment from the shaft **I**. In consequence, it can be written
►for the real system

$$M_M = k_1\left(\varphi_1 - \varphi_2\right) \quad (24)$$

►for the system with equivalent rigidity

$$M_{Meqv} = k_{1eqv}\left(\varphi_1^* - \varphi_2^*\right) \quad (25)$$

Dividing the relations (24) and (25) it is obtained

$$\frac{\varphi_1^* - \varphi_2^*}{\varphi_1 - \varphi_2} = \frac{k_1}{k_{1eqv}} \frac{M_{Meqv}}{M_M} \quad (26)$$

Considering the fraction of the rigidities done by (23), the relation (26) becomes

$$\sqrt{\frac{k_1}{k_{1eqv}}} = \frac{k_1}{k_{1eqv}} \frac{M_{Meqv}}{M_M}, \quad (27)$$

or

$$\frac{k_{1eqv}}{k_1} = \left(\frac{M_{Meqv}}{M_M}\right)^2 \quad (28)$$

From the relation (28),we may say that the equivalent rigidity of the shaft **I** is function of the fraction of the motor moments as follows:

$$k_{1eqv} = k_1\left(\frac{M_{Meqv}}{M_M}\right)^2 \quad (29)$$

The fraction between motor moments from (29) can be determined by writing the motor power both for real system and for equivalent system.

a)Gearing with no power losses ($\eta = 1$)

Considering **2-3** as ideal, all power from the motor goes to the working device, that's why we may write:

$$P = M_M\omega_1 = M_{Meqv}\omega_4 \quad (30)$$

From (30), we can write:

$$\frac{M_{Meqv}}{M_M} = \frac{\omega_1}{\omega_4} \quad (31)$$

Since $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$, the fraction between the motor moments can be written function of the gear ratio as follows:

$$\frac{M_{Meqv}}{M_M} = \frac{\omega_1}{\omega_4} = \frac{\omega_2}{\omega_3} = i \quad (32)$$

In this case, the calculus formula of the equivalent rigidity of the driving shaft on the driven shaft axle is:

$$k_{1eqv} = k_1 i^2 \quad (33)$$

a)Gearing with power losses ($\eta < 1$)

Taking into consideration the mechanical losses from the gearing **2-3**, the power from the motor goes partially to the working device and the difference is dissipated in the gearing. In this case, the balance of the power being can be written

$$P = M_M\omega_1 = M_{Meqv}\omega_4 - P_{loss}, \quad (34)$$

where P_{loss} is the power losses in the gearing.

Writing the loss of power as a function of ω_4 like

$$P_{loss} = M_{loss}\omega_4 \quad (35)$$

where M_{loss} is the equivalent loss of moment, **<u>in the steady-state conditions</u>**, the power balance done by (34) can be written like

$$M_M\omega_1 = \left(M_{Meqv} - M_{loss}\right)\omega_4, \quad (36)$$

or

$$\frac{M_{Meqv} - M_{loss}}{M_{Meqv}} = \frac{M_M}{M_{Meqv}} \frac{\omega_2}{\omega_3} \quad (37)$$

Since, in the left side is the mechanical efficiency of the gearing and the fraction between angular speeds from right side is the gearing ratio, the relation (37) becomes

$$\frac{M_{Meqv}}{M_M} = \frac{i}{\eta} \tag{38}$$

With the determinated fraction of the moments (38), we can write the calculus formula for the equivalent rigidity of the driving shaft with real gearing like a function of the ratio and the mechanical efficiency as follows:

$$k_{1eqv} = k_1 \frac{i^2}{\eta^2} \tag{39}$$

3. The equivalent rigidities of the mechanism's shafts with gearings

To exemplify the method of the rigidity equation for the mechanism with multiple gearing, it is considered the driving system for a belt conveyor from fig. 4. The skeleton diagram of the acting device is shown in fig. 5, where **2**, **3**, **4**, **5**, **6** and **7** are the gearing wheels of the mechanical transmission. It considers as known the mechanical efficiencies and the ratio of the gearings as follows:

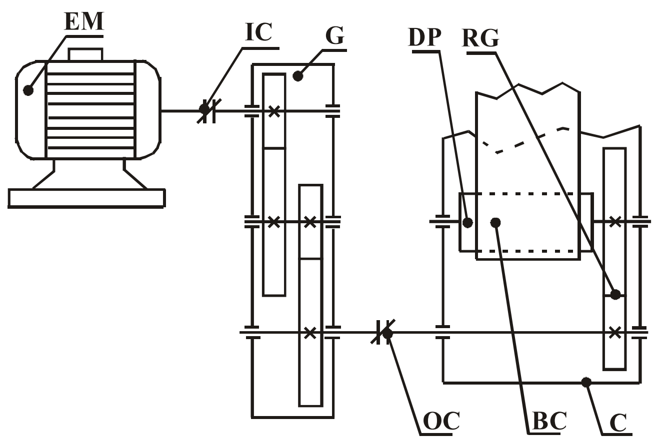

Fig. 4 The principle model of a belt conveyor
Legend: **EM**-electromotor, **IC**-inside coupling, **OC**-outside coupling, **G**-gear reducer unit, **C**-case, **RG**-reducing gear, **BC**-belt conveyor, **DP**-drive pulley

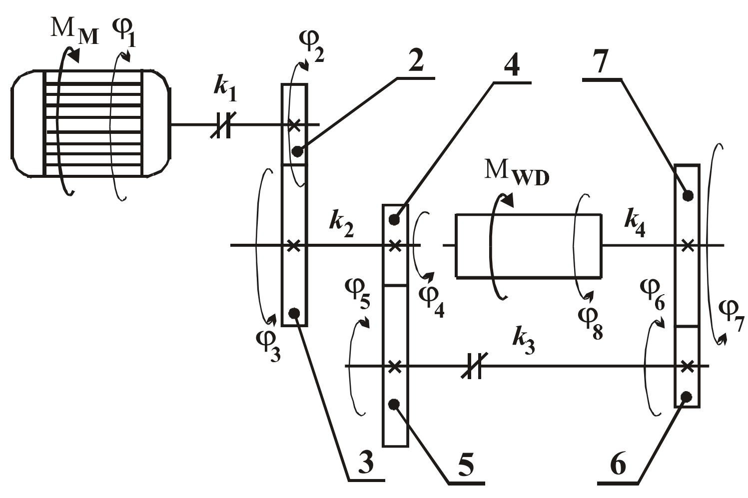

Fig. 5 The skeleton diagram for the belt conveyor

-gearing **2-3** → η_1, i_1
-gearing **4-5** → η_2, i_2
-gearing **6-7** → η_3, i_3

Using the calculus relationships

determinated in §2, we will equate the shafts' rigidities both on the electromotor axle and on the drive pulley axle.

3.1. Rigidities on the electromotor axle

Figure 6 shows the calculus diagram of the equivalent rigidities on electromotor axle, where the formulae for the equivalent moments of inertia can be taken from [2]. The calculus relationships used in this case are (14) for ideal gearings and (20) for real gearings.

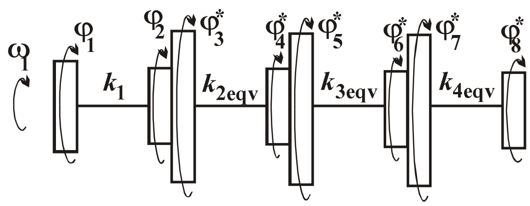

Fig. 6 The calculus diagram for the equivalent rigidities on the electromotor axle shaft

The equivalent rigidities on the electromotor axle for the shafts 2, 3 and 4 are shown in the table 1 for both cases (without and with mechanical losses).

Table 1 Equivalent rigidities on the driving axle shaft

Ideal gearings ($\eta = 1$)	Real gearings ($\eta < 1$)
$k_{2eqv} = \frac{k_2}{i_1^2}$	$k_{2eqv} = \frac{k_2}{i_1^2 \eta_1^2}$
$k_{3eqv} = \frac{k_3}{i_1^2 i_2^2}$	$k_{3eqv} = \frac{k_3}{i_1^2 i_2^2 \eta_1^2 \eta_2^2}$
$k_{4eqv} = \frac{k_4}{i_1^2 i_2^2 i_3^2}$	$k_{4eqv} = \frac{k_4}{i_1^2 i_2^2 i_3^2 \eta_1^2 \eta_2^2 \eta_3^2}$

3.2. Rigidities on the drive pulley axle

Figure 7 shows the calculus diagram of the equivalent rigidities on the drive axle shaft, where, like for §3.1, the equivalent moments of inertia can be taken from [2]. The calculus relationships used in this case are (33) for ideal gearings and (39) for real gearings.

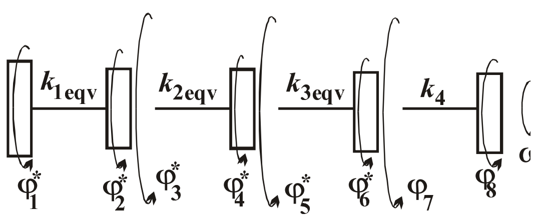

Fig. 7 The calculus diagram for the equivalent rigidities on the drive pulley axle shaft

Table 2 shows the equivalent rigidities on the drive pulley axle for the shafts 1, 2 and 3, both in the case when the mechanical losses are taken into consideration and in case they aren't.

5. Conclusions

The methods and calculus formulae presented in this study are useful both to design engineers and to dynamics experts, as well as to students, , candidates for master's and doctor's degree.

Regarding the ratio i of the gearings, we may draw some conclusions about its influence on the equivalent rigidities:

1^0if $i = 1$ (gearing for changing the sense of rotation only), the equivalent rigidities stay unchanged;

2^0if $i > 1$ (reduced step gearing), the rigidity of the driving shaft (on the driven shaft axle) is amplified by i^2 and the rigidity of the driven shaft (on the driving shaft axle) is divided by i^2;

3^0if $i < 1$ (amplifier step gearing) the rigidity of the driving shaft (on the driven shaft axle) is divided by i^2 and the rigidity of the driven shaft (on the driving shaft axle) is amplified by i^2.

Taking into consideration the losses of power in the gearings trough their mechanical coefficients $\eta < 1$, the equivalent rigidities of the shafts are always increased by multiplying with $\frac{1}{\eta^2}$.

Table 2 Equivalent rigidities on the driven axle shaft

Ideal gearings ($\eta = 1$)	Real gearings ($\eta < 1$)
$k_{1eqv} = k_1 i_3^2 i_2^2 i_1^2$	$k_{1eqv} = k_1 \frac{i_3^2 i_2^2 i_1^2}{\eta_3^2 \eta_2^2 \eta_1^2}$
$k_{2eqv} = k_2 i_3^2 i_2^2$	$k_{2eqv} = k_2 \frac{i_3^2 i_2^2}{\eta_3^2 \eta_2^2}$
$k_{3eqv} = k_3 i_3^2$	$k_{3eqv} = k_3 \frac{i_3^2}{\eta_3^2}$

5. References

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